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A Study on a Solvable Two-Dimensional System of Fourth-Order Difference Equations

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ABSTRACT

Difference equations play a crucial role in modeling discrete dynamical systems across a wide range of scientific disciplines, including biology, economics, engineering, and physics. In this article, we examine a nonlinear system of rational difference equations of order four, defined by

$$\begin{cases} x_{n+1} = \dfrac{y_n y_{n-3}}{a x_{n-2} - b y_{n-3}}, \\ y_{n+1} = \dfrac{x_n x_{n-3}}{c y_{n-2} - d y_{n-3}}, \end{cases} \qquad n \in \mathbb{N}_0,$$

where a, b, c, and d are real parameters. The system is initialized with nonzero real values for $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0$. The objectives of this study are to derive explicit closed-form solutions, determine conditions for the existence and uniqueness of solutions, and analyze the qualitative behavior of the system, including stability and asymptotic properties. The results offer new insights into the intricate dependence of the system's dynamics on both the parameters and the initial conditions. This contributes to a deeper understanding of high-order nonlinear difference systems and their complex behaviors.

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1. Introduction

Rational difference equations are a class of discrete dynamical systems in which each term evolves based on ratios of previous terms. These equations frequently arise in scientific and engineering contexts where system changes depend on proportional relationships, such as in population dynamics, economic models, control theory, and biological interactions [1]–[6]. Their nonlinear and

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often complex behavior has made them a subject of extensive study, particularly with regard to stability, boundedness, periodicity, and chaos [7]–[12]. Moreover, rational difference equations and their associated systems play a fundamental role in modeling diverse real-world phenomena across disciplines including physics, chemistry, sociology, and economics [13]–[25]. A comprehensive understanding of their qualitative dynamics is essential for accurately predicting the long-term behavior of such systems in both the natural and social sciences.

The quest for closed-form solutions to difference equations has a long history, dating back over three centuries. Renowned mathematicians such as de Moivre [26], Euler [27], Lagrange [28], and Laplace [29] made early contributions to this field and laid the foundation for finding closed-form formulas for solutions to difference equations. Over the years, various methods have been developed to solve both linear and nonlinear difference equations, with constant coefficients (see. e.g., [30]–[52] and references cited therein).

In recent decades, there has been a growing interest in the solvability theory of difference equations, driven in part by advancements in computational systems. However, reliance solely on computer-assisted methods has led to some issues, including the reproduction of known results without providing theoretical explanations. This problem was first highlighted in 2004; [53] when a constructive method was successfully employed to solve a particular nonlinear difference equation. Moreover, concrete systems of difference equations have gained popularity in recent years, thanks to the work of researchers like Papaschinopoulos and Schinas [54], [55]. Motivated by their contributions, researchers have focused their efforts on understanding and solving these concrete systems. Furthermore, there has been significant interest in representing solutions to difference equations in terms of specific sequences, such as Fibonacci sequences.

This paper aims to address these challenges by providing closed-form solutions for a class of difference equations. Specifically, we consider the extension of a system of difference equations and present elegant and constructive closed-form formulas for their solutions. The findings of this study contribute to the understanding of difference equations and their applications in various scientific disciplines. The primary objective of this study is to introduce a closed-form solution and investigate the qualitative behavior of a specific difference equation:

$$x_{n+1} = \frac{y_n y_{n-3}}{a x_{n-2} - b y_{n-3}}, \quad y_{n+1} = \frac{x_n x_{n-3}}{c y_{n-2} - d x_{n-3}}, \quad n \in \mathbb{N}_0. \tag{1}$$

By providing analytical expressions for the solutions and analyzing their qualitative behavior, this study aims to contribute to a deeper understanding of difference equations and their applications in various scientific fields.

The nonlinear system under investigation not only presents significant theoretical challenges, but also has the potential to model real-world phenomena characterized by delayed feedback, such as population dynamics, economic cycles, and control systems. Its fourth-order structure and coupled rational form make it well suited to capture complex temporal dependencies. Key contributions of this paper include the derivation of explicit solution formulas for the system and numerical simulations that illustrate and validate the theoretical findings. These results improve our understanding of the dynamics of high-order nonlinear rational systems and provide a foundation for future research on more general or applied models.

2. Main Results

In this section, we investigate the solution behavior of the following nonlinear system of fourth-order rational difference equations:

$$x_{n+1} = \frac{y_n y_{n-3}}{a x_{n-2} - b y_{n-3}}, \; y_{n+1} = \frac{x_n x_{n-3}}{c y_{n-2} - d x_{n-3}}, \quad n \in \mathbb{N}_0, \tag{2}$$

Where a, b, c, and d are real parameters, and the initial conditions

$$x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1} \text{ and } y_0$$

are arbitrary nonzero real numbers chosen to ensure that the denominators in (2) are nonzero for all n; specifically,

$$ax_{n-2} - by_{n-3} \neq 0, \quad cy_{n-2} - dx_{n-3} \neq 0.$$

To simplify the system and facilitate analytical solutions, we introduce the transformation:

$$u_n = \frac{y_{n-1}}{x_n}, \quad v_n = \frac{x_{n-1}}{y_n}, \tag{3}$$

With corresponding initial values:

$$\begin{aligned} u_0 &= \frac{y_{-1}}{x_0}, & v_0 &= \frac{x_{-1}}{y_0}, \\ u_{-1} &= \frac{y_{-2}}{x_{-1}}, & v_{-1} &= \frac{x_{-2}}{y_{-1}}, \\ u_{-2} &= \frac{y_{-3}}{x_{-2}}, & v_{-2} &= \frac{x_{-3}}{y_{-2}}. \end{aligned} \tag{4}$$

This transformation reduces the original system to a more tractable form, allowing for a detailed investigation of its dynamics. System (2) can thus be expressed as:

$$u_{n+1} = \frac{a - bu_{n-2}}{u_{n-2}}, \quad v_{n+1} = \frac{c - dv_{n-2}}{v_{n-2}}, \quad n \in \mathbb{N}_0 \tag{5}$$

Let

$$u_m^j = u_{3m-1}, \quad v_m^j = v_{3m-1}, \tag{6}$$

for $m \geq -2$, $j = 0, 1, 2$. Here, to give a closed form for the well defined solutions of system (5), We consider the system of two difference equations nonlinear first-order.

$$u_{n+1}^j = \frac{a - bu_{n-2}^j}{u_{n-2}^j}, \quad v_{n+1}^j = \frac{c - dv_{n-2}^j}{v_{n-2}^j}, \quad n \in \mathbb{N}_0 \tag{7}$$

for $m \geq -2$, $j = 0, 1, 2$. Let

$$u_n^j = \frac{z_n}{z_{n-1}}, \quad v_n^j = \frac{w_n}{w_{n-1}}, \tag{8}$$

$$z_0 = u_0^j,\ z_{-1} = 1,\ w_0 = v_0^j,\ and\, w_{-1} = 1.$$

for $m \geq -2$, $j = 0, 1, 2$. Thus, we get following two linear difference equations

$$z_{n+1} + bz_n - az_{n-1} = 0, \tag{9}$$

$$w_{n+1} + dw_n - cw_{n-1} = 0, \tag{10}$$

With initial condition z_{-1}, z_0, w_{-1}, $w_0 \in \mathbb{C}$.
Let λ_1 and λ_2 be the zeros of the characteristic polynomial $P_z(\lambda) = \lambda^2 + b\lambda - a, \Delta = b^2 + 4a$ or $(P_w(\lambda) = \lambda^2 + b\lambda - c, \Delta = d^2 + 4c)$, where

$$\begin{array}{ll} \lambda_1 \neq \lambda_2 \in \mathbb{R} & if \Delta > 0, \\ \lambda_1 = \lambda_2 & if \Delta = 0, \\ \lambda_1 \neq \lambda_2 \in \mathbb{C} & if \Delta < 0, \end{array}$$

From the above and also using the article [56] the lemma are correct

Lemma 2.1 the solution to Equation (9) and (10) can be written as follows:

1. if $\Delta_{z,w} > 0$
   - the solution to Equation (9) , Then

$$z_n = z_0 \phi^{(n+1)} + a z_{-1} \phi^n , \tag{11}$$

   Where

$$\phi = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}, \ \lambda_\pm = \frac{-b \pm \sqrt{b^2+4a}}{2}.$$

   - the solution to Equation (10) , Then

$$z_n = z_0 \psi^{(n+1)} + c z_{-1} \psi^n , \tag{12}$$

   Where

$$\psi = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}, \ \lambda_\pm = \frac{-d \pm \sqrt{d^2+4c}}{2}$$

2. if $\Delta_{z,w} = 0$
   - the solution to Equation (9), Then

$$z_n = z_0 (n+1) \lambda_b^n - z_{-1} n \lambda_b^{n+1} , \ where \ \lambda_b = \frac{-b}{2} \tag{13}$$

   - the solution to Equation (10), Then

$$z_n = z_0 (n+1) \lambda_d^n - z_{-1} n \lambda_d^{n+1} , \ where \ \lambda_d = \frac{-d}{2} \tag{14}$$

3. if $\Delta_{z,w} < 0$
   - the solution to Equation (9), Then

$$z_n = \sqrt{a} z_{-1} \sin n\alpha_1 - z_0 \sin (n+1) \alpha_1 , \tag{15}$$

   where

$$\alpha_1 = \arctan \left(\frac{\sqrt{-b^2-4a}}{-b} \right)$$

   .

   - the solution to Equation (10), Then

$$w_n = \sqrt{c} w_{-1} \sin n\alpha_2 - w_0 \sin (n+1) \alpha_2 , \tag{16}$$

   where

$$\alpha_2 = \arctan \left(\frac{\sqrt{-d^2-4c}}{-d} \right).$$

where $z_{-1}, z_0 \in \mathbb{R}$ are the initial conditions.

From (8) and lemma2.1 it follows that

$$\begin{aligned} u_n^j &= \frac{a\phi_n + \phi_{n+1}u_0^j}{a\phi_{n-1} + \phi_n u_0^j}, \\ v_n^j &= \frac{c\psi_n + \psi_{n+1}u_0^j}{c\psi_{n-1} + \psi_n u_0^j}, \end{aligned} \tag{17}$$

for $j = 0, 1, 2$. Using (6), we obtain

$$\begin{aligned} u_{3n-j} &= \frac{a\phi_n + \phi_{n+1}u_{-j}}{a\phi_{n-1} + \phi_n u_{-j}}, \\ v_{3n-j} &= \frac{c\psi_n + \psi_{n+1}u_{-j}}{c\psi_{n-1} + \psi_n u_{-j}}, \end{aligned} \tag{18}$$

for $j = 0, 1, 2$,that is

$$\left\{ \begin{aligned} u_{3n} &= \tfrac{a\phi_n+\phi_{n+1}u_0}{a\phi_{n-1}+\phi_n u_0}, \\ u_{3n-1} &= \tfrac{a\phi_n+\phi_{n+1}u_{-1}}{a\phi_{n-1}+\phi_n u_{-1}}, \\ u_{3n-2} &= \tfrac{a\phi_n+\phi_{n+1}u_{-2}}{a\phi_{n-1}+\phi_n u_{-2}}, \end{aligned} \right. \quad \text{and} \quad \left\{ \begin{aligned} v_{3n} &= \tfrac{c\psi_n+\psi_{n+1}v_0}{c\psi_{n-1}+\psi_n v_0}, \\ v_{3n-1} &= \tfrac{c\psi_n+\psi_{n+1}v_{-1}}{c\psi_{n-1}+\psi_n v_{-1}}, \\ v_{3n-2} &= \tfrac{c\psi_n+\psi_{n+1}v_{-2}}{c\psi_{n-1}+\psi_n v_{-2}}, \end{aligned} \right. \tag{19}$$

if $(\Delta_{z,w} > 0)$.

$$\left\{ \begin{aligned} u_{3n} &= \tfrac{n\lambda_b^2-(n+1)\lambda_b u_0}{(n-1)\lambda_b - nu_0}, \\ u_{3n-1} &= \tfrac{n\lambda_b^2-(n+1)\lambda_b u_{-1}}{(n-1)\lambda_b - nu_{-1}}, \\ u_{3n-2} &= \tfrac{n\lambda_b^2-(n+1)\lambda_b u_{-2}}{(n-1)\lambda_b - nu_{-2}}, \end{aligned} \right. \quad \text{and} \quad \left\{ \begin{aligned} v_{3n} &= \tfrac{n\lambda_d^2-(n+1)\lambda_d v_0}{(n-1)\lambda_d - nv_0}, \\ v_{3n-1} &= \tfrac{n\lambda_d^2-(n+1)\lambda_d v_{-1}}{(n-1)\lambda_d - nv_{-1}}, \\ v_{3n-2} &= \tfrac{n\lambda_d^2-(n+1)\lambda_d v_{-2}}{(n-1)\lambda_d - nv_{-2}}, \end{aligned} \right. \tag{20}$$

if $(\Delta_{z,w} = 0)$. And

$$\left\{ \begin{aligned} u_{3n} &= \tfrac{\sqrt{a}\sin n\alpha_1 + u_0 \sin(n+1)\alpha_1}{\sqrt{a}\sin(n-1)\alpha_1 + u_0 \sin n\alpha_1}, \\ u_{3n-1} &= \tfrac{\sqrt{a}\sin n\alpha_1 + u_{-1} \sin(n+1)\alpha_1}{\sqrt{a}\sin(n-1)\alpha_1 + u_{-1} \sin n\alpha_1}, \\ u_{3n-2} &= \tfrac{\sqrt{a}\sin n\alpha_1 + u_{-2} \sin(n+1)\alpha_1}{\sqrt{a}\sin(n-1)\alpha_1 + u_{-2} \sin n\alpha_1}, \end{aligned} \right. \quad \text{and} \quad \left\{ \begin{aligned} v_{3n} &= \tfrac{\sqrt{c}\sin n\alpha_2 + v_0 \sin(n+1)\alpha_2}{\sqrt{c}\sin(n-1)\alpha_1 + v_0 \sin n\alpha_2}, \\ v_{3n-1} &= \tfrac{\sqrt{c}\sin n\alpha_2 + v_{-1} \sin(n+1)\alpha_2}{\sqrt{c}\sin(n-1)\alpha_1 + v_{-1} \sin n\alpha_2}, \\ v_{3n-2} &= \tfrac{\sqrt{c}\sin n\alpha_2 + v_{-2} \sin(n+1)\alpha_2}{\sqrt{c}\sin(n-1)\alpha_1 + v_{-2} \sin n\alpha_2}, \end{aligned} \right. \tag{21}$$

if $(\Delta_{z,w} < 0)$.
Let

$$x_n = \frac{y_{n-1}}{u_n}, \tag{22}$$

$$y_n = \frac{x_{n-1}}{v_n}. \tag{23}$$

Using formulas (22) and (23), after some calculations, we obtain

$$x_{6n} = \frac{x_{6n-6}}{u_{6n}v_{6n-1}u_{6n-2}v_{6n-3}u_{6n-4}u_{6n-5}}, \tag{24}$$

$$y_{6n}=\frac{y_{6n-6}}{v_{6n}u_{6n-1}v_{6n-2}u_{6n-3}v_{6n-4}u_{6n-5}}, \tag{25}$$

for any $n\in\mathbb{N}$. Multiplying the equalities (24) and (25) from 1 to n, respectively, it follows that

$$x_{6n} = \frac{x_0}{\prod\limits_{i=1}^{n}\left(u_{6i}v_{6i-1}u_{6i-2}v_{6i-3}u_{6i-4}u_{6i-5}\right)}, \tag{26}$$

$$y_{6n} = \frac{y_0}{\prod\limits_{i=1}^{n}\left(v_{6i}u_{6i-1}v_{6i-2}u_{6i-3}v_{6i-4}u_{6i-5}\right)}. \tag{27}$$

Thus substitution of Equations (26) and (27) into Equations (22) and (23) leads to

$$\begin{aligned}
x_{6n-1} &= v_{6n}y_{6n}=\frac{y_0v_{6n}}{\prod\limits_{i=1}^{n}\left(v_{6i}u_{6i-1}v_{6i-2}u_{6i-3}v_{6i-4}u_{6i-5}\right)},\\
y_{6n-1} &= u_{6n}x_{6n}=\frac{x_0u_{6n}}{\prod\limits_{i=1}^{n}\left(u_{6i}v_{6i-1}u_{6i-2}v_{6i-3}u_{6i-4}u_{6i-5}\right)},\\
x_{6n-2} &= v_{6n-1}y_{6n-1}=\frac{x_0u_{6n}v_{6n-1}}{\prod\limits_{i=1}^{n}\left(u_{6i}v_{6i-1}u_{6i-2}v_{6i-3}u_{6i-4}u_{6i-5}\right)},\\
y_{6n-2} &= u_{6n-1}x_{6n-1}=\frac{y_0v_{6n}u_{6n-1}}{\prod\limits_{i=1}^{n}\left(v_{6i}u_{6i-1}v_{6i-2}u_{6i-3}v_{6i-4}u_{6i-5}\right)},\\
x_{6n-3} &= v_{6n-2}y_{6n-2}=\frac{y_0v_{6n}u_{6n-1}v_{6n-2}}{\prod\limits_{i=1}^{n}\left(v_{6i}u_{6i-1}v_{6i-2}u_{6i-3}v_{6i-4}u_{6i-5}\right)},\\
y_{6n-3} &= u_{6n-2}x_{6n-2}=\frac{x_0u_{6n}v_{6n-1}u_{6n-2}}{\prod\limits_{i=1}^{n}\left(u_{6i}v_{6i-1}u_{6i-2}v_{6i-3}u_{6i-4}u_{6i-5}\right)},\\
x_{6n+1} &= \frac{y_{6n}}{u_{6n+1}}=\frac{y_0}{u_{6n+1}\prod\limits_{i=1}^{n}\left(v_{6i}u_{6i-1}v_{6i-2}u_{6i-3}v_{6i-4}u_{6i-5}\right)},\\
y_{6n+1} &= \frac{x_{6n}}{v_{6n+1}}=\frac{x_0}{v_{6n+1}\prod\limits_{i=1}^{n}\left(u_{6i}v_{6i-1}u_{6i-2}v_{6i-3}u_{6i-4}u_{6i-5}\right)},\\
x_{6n+2} &= \frac{y_{6n+1}}{u_{6n+2}}=\frac{x_0}{v_{6n+1}u_{6n+2}\prod\limits_{i=1}^{n}\left(u_{6i}v_{6i-1}u_{6i-2}v_{6i-3}u_{6i-4}u_{6i-5}\right)},\\
y_{6n+2} &= \frac{x_{6n+1}}{v_{6n+2}}=\frac{y_0}{u_{6n+1}v_{6n+2}\prod\limits_{i=1}^{n}\left(v_{6i}u_{6i-1}v_{6i-2}u_{6i-3}v_{6i-4}u_{6i-5}\right)}.
\end{aligned}$$

Let's put together the following two relationships

$$\begin{aligned}
\kappa_i &= v_{6i}u_{6i-1}v_{6i-2}u_{6i-3}v_{6i-4}u_{6i-5},\\
\chi_i &= u_{6i}v_{6i-1}u_{6i-2}v_{6i-3}u_{6i-4}u_{6i-5}.
\end{aligned} \tag{28}$$

Using these relationships (19), (19), (19) and taking into account that

$$u_0=\frac{y_{-1}}{x_0},\, v_0=\frac{x_{-1}}{y_0},\, u_{-1}=\frac{y_{-2}}{x_{-1}},\, v_{-1}=\frac{x_{-2}}{y_{-1}},\, u_{-2}=\frac{y_{-3}}{x_{-2}}\, and\, v_{-2}=\frac{x_{-3}}{y_{-2}}.$$

From the above consideration we see that the following result holds.

Corollary 2.2 *Let $\{x_n, y_n\}_{n\geq -1}$ be a solution to the system of difference equations of four order of the form*

$$x_{n+1} = \frac{y_n y_{n-3}}{a x_{n-2} - b y_{n-3}}, \; y_{n+1} = \frac{x_n x_{n-3}}{c y_{n-2} - d x_{n-3}}$$

is

$$x_{6n-3} = y_0 v_{6n} u_{6n-1} v_{6n-2} \left[\prod_{i=1}^{n} \kappa_i\right]^{-1}, \; x_{6n-2} = x_0 u_{6n} v_{6n-1} \left[\prod_{i=1}^{n} \chi_i\right]^{-1}, \; x_{6n-1} = y_0 v_{6n} \left[\prod_{i=1}^{n} \kappa_i\right]^{-1},$$

$$x_{6n} = x_0 \left[\prod_{i=1}^{n} \chi_i\right]^{-1}, \; x_{6n+1} = \frac{y_0}{u_{6n+1}} \left[\prod_{i=1}^{n} \kappa_i\right]^{-1}, \; x_{6n+2} = \frac{x_0}{v_{6n+1} u_{6n+2}} \left[\prod_{i=1}^{n} \chi_i\right]^{-1}.$$

And

$$y_{6n-3} = x_0 u_{6n} v_{6n-1} u_{6n-2} \left[\prod_{i=1}^{n} \chi_i\right]^{-1}, \; y_{6n-2} = y_0 v_{6n} u_{6n-1} \left[\prod_{i=1}^{n} \kappa_i\right]^{-1}, \; y_{6n-1} = x_0 u_{6n} \left[\prod_{i=1}^{n} \chi_i\right]^{-1},$$

$$y_{6n} = y_0 \left[\prod_{i=1}^{n} \kappa_i\right]^{-1}, \; y_{6n+1} = \frac{x_0}{v_{6n+1}} \left[\prod_{i=1}^{n} \chi_i\right]^{-1}, \; y_{6n+2} = \frac{y_0}{u_{6n+1} v_{6n+2}} \left[\prod_{i=1}^{n} \kappa_i\right]^{-1}.$$

with

$$u_0 = \frac{y_{-1}}{x_0}, v_0 = \frac{x_{-1}}{y_0}, u_{-1} = \frac{y_{-2}}{x_{-1}}, v_{-1} = \frac{x_{-2}}{y_{-1}}, u_{-2} = \frac{y_{-3}}{x_{-2}} \mathit{and} v_{-2} = \frac{x_{-3}}{y_{-2}}.$$

1. *If* $\Delta_z > 0$ *and* $\Delta_w > 0$, *to obtain*

$$\kappa_i = \left[\frac{c\psi_{2n} + \psi_{2n+1} v_0}{c\psi_{2n-1} + \psi_{2n} v_0} \cdot \frac{a\phi_{2n} + \phi_{2n+1} u_{-1}}{a\phi_{2n-1} + \phi_{2n} u_{-1}} \cdot \frac{c\psi_{2n} + \psi_{2n+1} v_{-2}}{c\psi_{2n-1} + \psi_{2n} v_{-2}} \cdot \right.$$
$$\left. \frac{a\phi_{2n-1} + \phi_{2n} u_0}{a\phi_{2n-2} + \phi_{2n-1} u_0} \cdot \frac{c\psi_{2n-1} + \psi_{2n} v_{-1}}{c\psi_{2n-2} + \psi_{2n-1} v_{-1}} \cdot \frac{a\phi_{2n-1} + \phi_{2n} u_{-2}}{a\phi_{2n-2} + \phi_{2n-1} u_{-2}}\right],$$

$$\chi_i = \left[\frac{a\phi_{2n} + \phi_{2n+1} v_0}{a\phi_{2n-1} + \phi_{2n} u_0} \cdot \frac{c\psi_{2n} + \psi_{2n+1} v_{-1}}{c\psi_{2n-1} + \psi_{2n} v_{-1}} \cdot \frac{a\phi_{2n} + \phi_{2n+1} u_{-2}}{a\phi_{2n-1} + \phi_{2n} u_{-2}} \cdot \right.$$
$$\left. \frac{c\psi_{2n-1} + \psi_{2n} v_0}{a\psi_{2n-2} + \psi_{2n-1} v_0} \cdot \frac{a\phi_{2n-1} + \phi_{2n} u_{-1}}{a\phi_{2n-2} + \phi_{2n-1} u_{-1}} \cdot \frac{c\psi_{2n-1} + \psi_{2n} v_{-2}}{c\psi_{2n-2} + \psi_{2n-1} v_{-2}}\right].$$

where

$$\phi = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}, \; \lambda_\pm = \frac{-b \pm \sqrt{b^2 + 4a}}{2}.$$

And

$$\psi = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}, \; \lambda_\pm = \frac{-d \pm \sqrt{d^2 + 4c}}{2}.$$

2. *If* $\Delta_z > 0$ *and* $\Delta_w = 0$

$$\kappa_i = \left[\frac{2n\lambda_d^2 - (2n+1)\lambda_d v_0}{(2n-1)\lambda_d - 2n v_0} \cdot \frac{a\phi_{2n} + \phi_{2n+1} u_{-1}}{a\phi_{2n-1} + \phi_{2n} u_{-1}} \cdot \frac{2n\lambda_d^2 - (2n+1)\lambda_d v_{-2}}{(2n-1)\lambda_d - 2n v_{-2}} \cdot \right.$$
$$\left. \frac{a\phi_{2n-1} + \phi_{2n} u_0}{a\phi_{2n-2} + \phi_{2n-1} u_0} \cdot \frac{(2n-1)\lambda_d^2 - 2n\lambda_d v_{-1}}{(2n-2)\lambda_d - (2n-1) v_{-1}} \cdot \frac{a\phi_{2n-1} + \phi_{2n} u_{-2}}{a\phi_{2n-2} + \phi_{2n-1} u_{-2}}\right],$$

$$\chi_i = \left[\frac{a\phi_{2n}+\phi_{2n+1}v_0}{a\phi_{2n-1}+\phi_{2n}u_0}\cdot\frac{2n\lambda_d^2-(2n+1)\lambda_d v_{-1}}{(2n-1)\lambda_d-2nv_{-1}}\cdot\frac{a\phi_{2n}+\phi_{2n+1}u_{-2}}{a\phi_{2n-1}+\phi_{2n}u_{-2}}\cdot\frac{(2n-1)\lambda_d^2-2n\lambda_d v_0}{(2n-2)\lambda_d-(2n-1)v_0}\cdot\frac{a\phi_{2n-1}+\phi_{2n}u_{-1}}{a\phi_{2n-2}+\phi_{2n-1}u_{-1}}\cdot\frac{(2n-1)\lambda_d^2-2n\lambda_d v_{-2}}{(2n-2)\lambda_d-(2n-1)v_{-2}}\right].$$

where

$$\lambda_d = \frac{-d}{2}.$$

3. *If* $\Delta_z > 0$ *and* $\Delta_w < 0$

$$\kappa_i = \left[\frac{\sqrt{c}\sin 2n\alpha_2+v_0\sin(2n+1)\alpha_2}{\sqrt{c}\sin(2n-1)\alpha_1+v_0\sin 2n\alpha_2}\cdot\frac{a\phi_{2n}+\phi_{2n+1}u_{-1}}{a\phi_{2n-1}+\phi_{2n}u_{-1}}\cdot\frac{\sqrt{c}\sin 2n\alpha_2+v_{-2}\sin(2n+1)\alpha_2}{\sqrt{c}\sin(2n-1)\alpha_1+v_{-2}\sin 2n\alpha_2}\cdot\frac{a\phi_{2n-1}+\phi_{2n}u_0}{a\phi_{2n-2}+\phi_{2n-1}u_0}\cdot\frac{\sqrt{c}\sin(2n-1)\alpha_2+v_{-1}\sin 2n\alpha_2}{\sqrt{c}\sin(2n-2)\alpha_1+v_{-1}\sin(2n-1)\alpha_2}\cdot\frac{a\phi_{2n-1}+\phi_{2n}u_{-2}}{a\phi_{2n-2}+\phi_{2n-1}u_{-2}}\right],$$

$$\chi_i = \left[\frac{a\phi_{2n}+\phi_{2n+1}v_0}{a\phi_{2n-1}+\phi_{2n}u_0}\cdot\frac{\sqrt{c}\sin 2n\alpha_2+v_0\sin(2n+1)\alpha_2}{\sqrt{c}\sin(2n-1)\alpha_1+v_{-1}\sin 2n\alpha_2}\cdot\frac{a\phi_{2n}+\phi_{2n+1}u_{-2}}{a\phi_{2n-1}+\phi_{2n}u_{-2}}\cdot\frac{\sqrt{c}\sin(2n-1)\alpha_2+v_0\sin 2n\alpha_2}{\sqrt{c}\sin(2n-2)\alpha_1+v_0\sin(2n-1)\alpha_2}\cdot\frac{a\phi_{2n-1}+\phi_{2n}u_{-1}}{a\phi_{2n-2}+\phi_{2n-1}u_{-1}}\cdot\frac{\sqrt{c}\sin(2n-1)\alpha_2+v_{-2}\sin 2n\alpha_2}{\sqrt{c}\sin(2n-2)\alpha_1+v_{-2}\sin(2n-1)\alpha_2}\right].$$

where

$$\alpha_2 = \arctan\left(\frac{\sqrt{-d^2-4c}}{-d}\right).$$

4. *If* $\Delta_z = 0$ *and* $\Delta_w = 0$

$$\kappa_i = \left[\frac{2n\lambda_d^2-(2n+1)\lambda_d v_0}{(2n-1)\lambda_d-2nv_0}\cdot\frac{2n\lambda_b^2-(2n+1)\lambda_b u_{-1}}{(2n-1)\lambda_b-2nu_{-1}}\cdot\frac{2n\lambda_d^2-(2n+1)\lambda_d v_{-2}}{(2n-1)\lambda_d-2nv_{-2}}\cdot\frac{(2n-1)\lambda_b^2-2n\lambda_b u_0}{(2n-2)\lambda_b-(2n-1)u_0}\cdot\frac{(2n-1)\lambda_d^2-2n\lambda_d v_{-1}}{(2n-2)\lambda_d-(2n-1)v_{-1}}\cdot\frac{(2n-1)\lambda_b^2-2n\lambda_b u_{-2}}{(2n-2)\lambda_b-(2n-1)u_{-2}}\right],$$

$$\chi_i = \left[\frac{2n\lambda_b^2-(2n+1)\lambda_b u_0}{(2n-1)\lambda_b-2nu_0}\cdot\frac{2n\lambda_d^2-(2n+1)\lambda_d v_{-1}}{(2n-1)\lambda_d-2nv_{-1}}\cdot\frac{2n\lambda_b^2-(2n+1)\lambda_b u_{-2}}{(2n-1)\lambda_b-2nu_{-2}}\cdot\frac{(2n-1)\lambda_d^2-2n\lambda_d v_0}{(2n-2)\lambda_d-(2n-1)v_0}\cdot\frac{(2n-1)\lambda_b^2-2n\lambda_b u_{-1}}{(2n-2)\lambda_b-(2n-1)u_{-1}}\cdot\frac{(2n-1)\lambda_d^2-2n\lambda_d v_{-2}}{(2n-2)\lambda_d-(2n-1)v_{-2}}\right].$$

where

$$\lambda_b = \frac{-b}{2},\ \lambda_d = \frac{-d}{2}.$$

5. *If* $\Delta_z = 0$ *and* $\Delta_w < 0$

$$\kappa_i = \left[\frac{\sqrt{c}\sin 2n\alpha_2+v_0\sin(2n+1)\alpha_2}{\sqrt{c}\sin(2n-1)\alpha_1+v_0\sin 2n\alpha_2}\cdot\frac{2n\lambda_b^2-(2n+1)\lambda_b u_{-1}}{(2n-1)\lambda_b-2nu_{-1}}\cdot\frac{\sqrt{c}\sin 2n\alpha_2+v_{-2}\sin(2n+1)\alpha_2}{\sqrt{c}\sin(2n-1)\alpha_1+v_{-2}\sin 2n\alpha_2}\cdot\frac{(2n-1)\lambda_b^2-2n\lambda_b u_0}{(2n-2)\lambda_b-(2n-1)u_0}\cdot\frac{\sqrt{c}\sin(2n+1)\alpha_2+v_{-1}\sin 2n\alpha_2}{\sqrt{c}\sin(2n-2)\alpha_1+v_{-1}\sin(2n-1)\alpha_2}\cdot\frac{(2n-1)\lambda_b^2-2n\lambda_b u_{-2}}{(2n-2)\lambda_b-(2n-1)u_{-2}}\right],$$

$$\chi_i = \left[\frac{2n\lambda_b^2-(2n+1)\lambda_b u_0}{(2n-1)\lambda_b-2nu_0}\cdot\frac{\sqrt{c}\sin 2n\alpha_2+v_{-1}\sin(2n+1)\alpha_2}{\sqrt{c}\sin(2n-1)\alpha_1+v_{-1}\sin 2n\alpha_2}\cdot\frac{2n\lambda_b^2-(2n+1)\lambda_b u_{-2}}{(2n-1)\lambda_b-2nu_{-2}}\cdot\frac{\sqrt{c}\sin(2n-1)\alpha_2+v_0\sin 2n\alpha_2}{\sqrt{c}\sin(2n-2)\alpha_1+v_0\sin(2n-1)\alpha_2}\cdot\frac{(2n-1)\lambda_b^2-2n\lambda_b u_{-1}}{(2n-2)\lambda_b-(2n-1)u_{-1}}\cdot\frac{\sqrt{c}\sin(2n-1)\alpha_2+v_{-2}\sin 2n\alpha_2}{\sqrt{c}\sin(2n-2)\alpha_1+v_{-2}\sin(2n-1)\alpha_2}\right].$$

where

$$\lambda_b = \frac{-b}{2}, \alpha_2 = \arctan\left(\frac{\sqrt{-d^2-4c}}{-d}\right).$$

6. *If* $\Delta_z < 0$ *and* $\Delta_w < 0$

$$\begin{aligned}\kappa_i = \ & \left[\frac{\sqrt{c}\sin 2n\alpha_2 + v_0 \sin(2n+1)\alpha_2}{\sqrt{c}\sin(2n-1)\alpha_1 + v_0 \sin 2n\alpha_2} \cdot \frac{\sqrt{a}\sin 2n\alpha_1 + u_{-1}\sin(2n+1)\alpha_1}{\sqrt{a}\sin(2n-1)\alpha_1 + u_{-1}\sin 2n\alpha_1} \cdot \right. \\ & \frac{\sqrt{c}\sin 2n\alpha_2 + v_{-2}\sin(2n+1)\alpha_2}{\sqrt{c}\sin(2n-1)\alpha_1 + v_{-2}\sin 2n\alpha_2} \frac{\sqrt{a}\sin(2n-1)\alpha_1 + u_0 \sin 2n\alpha_1}{\sqrt{a}\sin(2n-2)\alpha_1 + u_0 \sin(2n-1)\alpha_1} \cdot \\ & \left. \frac{\sqrt{c}\sin(2n+1)\alpha_2 + v_{-1}\sin 2n\alpha_2}{\sqrt{c}\sin(2n-2)\alpha_1 + v_{-1}\sin(2n-1)\alpha_2} \cdot \frac{\sqrt{a}\sin(2n-1)\alpha_1 + u_{-2}\sin 2n\alpha_1}{\sqrt{a}\sin(2n-2)\alpha_1 + u_{-2}\sin(2n-1)\alpha_1}\right],\end{aligned}$$

$$\begin{aligned}\chi_i = \ & \left[\frac{\sqrt{a}\sin 2n\alpha_1 + u_0 \sin(2n+1)\alpha_1}{\sqrt{a}\sin(2n-1)\alpha_1 + u_0 \sin 2n\alpha_1} \cdot \frac{\sqrt{c}\sin 2n\alpha_2 + v_{-1}\sin(2n+1)\alpha_2}{\sqrt{c}\sin(2n-1)\alpha_1 + v_{-1}\sin 2n\alpha_2} \cdot \right. \\ & \frac{\sqrt{a}\sin 2n\alpha_1 + u_{-2}\sin(2n+1)\alpha_1}{\sqrt{a}\sin(2n-1)\alpha_1 + u_{-2}\sin 2n\alpha_1} \cdot \frac{\sqrt{c}\sin(2n-1)\alpha_2 + v_0 \sin 2n\alpha_2}{\sqrt{c}\sin(2n-2)\alpha_1 + v_0 \sin(2n-1)\alpha_2} \cdot \\ & \left. \frac{\sqrt{a}\sin(2n-1)\alpha_1 + u_{-1}\sin 2n\alpha_1}{\sqrt{a}\sin(2n-2)\alpha_1 + u_{-1}\sin(2n-1)\alpha_1} \cdot \frac{\sqrt{c}\sin(2n-1)\alpha_2 + v_{-2}\sin 2n\alpha_2}{\sqrt{c}\sin(2n-2)\alpha_1 + v_{-2}\sin(2n-1)\alpha_2}\right].\end{aligned}$$

where

$$\alpha_1 = \arctan\left(\frac{\sqrt{-b^2-4a}}{-b}\right), \alpha_2 = \arctan\left(\frac{\sqrt{-d^2-4c}}{-d}\right).$$

Remark 2.3 *The periodic solutions of the* $1-$*dimensional rational difference equation*

$$x_{n+1} = \frac{x_n x_{n-3}}{a x_{n-2} - b x_{n-3}}, \ n \in \mathbb{N}_0,$$

can be obtained from system (1) by taking $x_{-j} = y_{-j}$, $j \in \{0, \ldots, 3\}$.

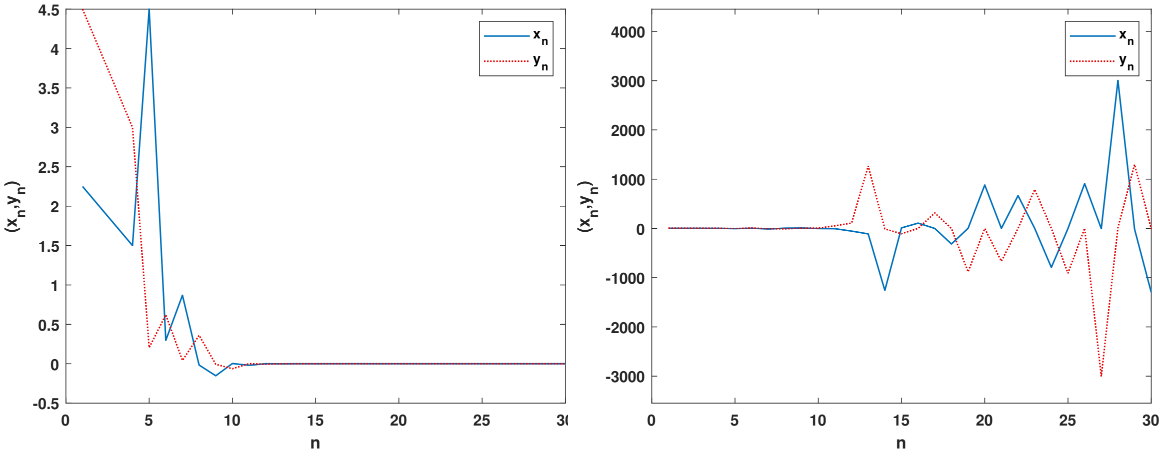
3. Numerical Simulation

To demonstrate the theoretical results, we perform numerical simulations of the system (1). Several parameter configurations are tested to explore different dynamical behaviors. The simulations provide visual insight into the system's stability, oscillation, and divergence patterns.

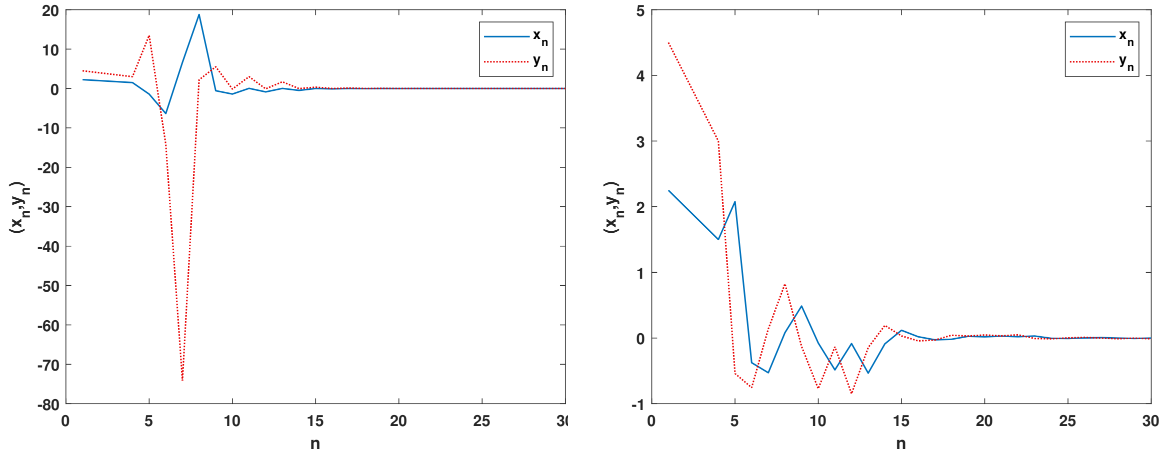
Example 3.1 To illustrate and validate the theoretical results, we perform numerical simulations of the system given in Equation (1) under the following parameter configurations:

- State 1: $a = -3$, $b = -2$, $c = 3$ and $d = -2$.
- State 2: $a = 1/4$, $b = 3/4$, $c = -3/5$ and $d = -2/5$.
- State 3: $a = 2$, $b = 3$, $c = 2/5$ and $d = 3/5$.
- State 4: $a = 1$, $b = -1$, $c = -1$ and $d = 1$.

The initial conditions are $x_{-k} = (6+k)/4$, $y_{-k} = (6+k)/2$, $k = 0, 1, 2, 3$. The diagram of the system (1) is shown in Fig. 1.



(a) State 1 **(b)** State 2



(c) State 3 **(d)** State 4

Fig. 1. Solutions of the system (1)

4. Conclusion

This study has presented closed-form solutions for a class of nonlinear rational difference equations, with particular focus on the system defined by Equation (1). Using a constructive approach, we derived explicit analytical expressions for the system's solutions and validated them through numerical simulations, confirming their accuracy and reliability.

Our findings contribute to the broader understanding of rational difference equations and highlight their applicability across various scientific domains. By analyzing both explicit solutions and the qualitative behavior of the system, this work advances theoretical knowledge in the study of high-order nonlinear systems.

In addition to addressing the two-dimensional system with constant coefficients, we have also derived well-defined closed-form solutions. This framework can be naturally extended to higher-dimensional systems, three-dimensional and beyond, incorporating either variable or constant coefficients as special cases.

Future research directions may include exploring generalized forms of the system in Equation

(1), investigating dynamics under varying conditions, and applying the proposed methodology to other classes of difference equations. Such studies would deepen insights into the qualitative behavior of complex discrete-time systems and foster the development of more effective analytical techniques.

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