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A Rigorous Stability Analysis of a Fractional-Order Anisotropic Diffusion Model for Edge-Preserving Biomedical Imaging

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ABSTRACT

Classical Perona-Malik models for image processing often suffer from mathematical ill-posedness and can introduce oversmoothing artifacts. This paper addresses these limitations by providing a rigorous theoretical analysis of a fractional-order (FO) anisotropic diffusion model. The primary contribution is the establishment of the model's well-posedness and stability properties, which have been underexplored. Using spectral decomposition and Lyapunov-energy methods, we prove the existence, uniqueness, and global Mittag-Leffler stability (MLS) of solutions under Neumann boundary conditions. This form of stability guarantees a predictable, algebraic decay to equilibrium, which is crucial for robust performance. Numerical simulations validate the theoretical framework and include a comparative analysis against the classical integer-order model. The results, evaluated with quantitative metrics, demonstrate the FO model's superior capability in preserving critical edge information while mitigating oversmoothing. By bridging rigorous mathematical theory with practical performance, this study provides a sound foundation for applying FO diffusion in biomedical imaging tasks, such as MRI denoising, where precision and stability are paramount.

1. Introduction

Perona and Malik introduced anisotropic diffusion (AD) in image processing, which preferentially smooths intra-region areas while preserving inter-region boundaries [1]. This technique has garnered significant interest due to its simplicity and effectiveness in image enhancement. [2] demonstrated that AD constitutes a steepest descent method for solving energy minimization problems, thereby explaining its dual mechanisms of smoothing and edge enhancement. However, the original Perona–Malik model, while numerically useful, is mathematically ill-posed due to discontinuities in its energy functional and the existence of dense global minima [3], [4]. This paradox has driven extensive research over two decades to develop well-posed models that retain the benefits of AD [5]. One promising direction is the use of FO derivatives, which generalize classical integer-order diffusion to capture non-local behaviors and provide a better balance between noise removal and edge preservation [6]–[11]. The governing equation for classical AD is given by:

$$\frac{\partial v(y,\tau)}{\partial \tau} = \nabla \cdot (\mathfrak{d}(y,\tau)\nabla v), \quad (y,\tau) \in \Omega \times \mathbb{R}_+^*, \tag{1}$$

where $v(y,\tau)$ represents image intensity and $\mathfrak{d}(y,\tau)$ is the diffusion coefficient that guides the smoothing process. By replacing the integer-order time derivative with a Caputo fractional derivative (CFD) of order $\beta \in (0,1)$, we obtain the FO diffusion model:

$$\begin{cases} {}_0^C D_\tau^\beta v(y,\tau) = \nabla \cdot (\mathfrak{d}(y,\tau)\nabla v), & (y,\tau) \in \Omega \times \mathbb{R}_+^*, \\ v(y,0) = v_0(y), & y \in \Omega, \\ \partial_\nu v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

While FO models have shown empirical success in reducing staircasing artifacts and improving denoising performance [12], [13], much of the existing literature focuses on algorithmic development and experimental results. A critical research gap remains in the rigorous mathematical analysis of these models. Specifically, the fundamental properties of existence, uniqueness, and long-term stability of solutions are often assumed rather than formally proven. This lack of a solid theoretical foundation limits the reliability and predictability of FO methods, especially for critical applications like biomedical imaging where stability guarantees are essential [14]–[23]. Recent investigations have advanced our understanding of FO systems. For instance, well-posedness for nonlocal diffusion was explored in [24], and various FO frameworks have been proposed [25], [26]. Stability, particularly global MLS, has been analyzed for related FO systems, including porous media problems [27], [28], reaction-diffusion systems [29]–[31], and systems with boundary control [32]–[42]. These studies provide valuable tools but have not been comprehensively applied to establish the stability of the foundational FO anisotropic diffusion model in (2).

This paper aims to bridge this theoretical gap. We provide a complete well-posedness and stability analysis for the FO anisotropic diffusion model. The main contributions of this research are as follows:

- We rigorously prove the existence and uniqueness of solutions for the FO model using spectral theory.
- We formally establish the Global MLS of the system using a Lyapunov-energy approach, providing a mathematical guarantee of convergence to a stable equilibrium.
- We validate our theoretical findings with numerical simulations and provide a quantitative comparison against the classical integer-order model to demonstrate the practical benefits of the FO approach in mitigating oversmoothing.
- We explicitly discuss the model's limitations and outline clear directions for future work, including extensions to 2D real-world images.

The paper is organized as follows: Section 2 introduces fundamental concepts from fractional calculus and MLS theory. Section 3 establishes well-posedness, stability, and energy decay properties of the FO model. Section 4 presents numerical simulations validating theoretical findings and demonstrating practical performance. Section 5 concludes with a discussion of future work and applications.

2. Preliminaries

Before developing our FO anisotropic diffusion model, we recall several fundamental concepts from fractional calculus and fractional stability theory that will be used in the sequel. Fractional integrals and derivatives generalize their integer-order counterparts by allowing non-integer orders of differentiation or integration; they provide a natural way to interpolate between purely diffusive (heat-like) and wave-like behaviors and to capture anomalous transport phenomena. The Riemann–Liouville fractional integral and the CFD are two of the most commonly used operators in this

setting. Moreover, solutions of FO differential equations are often expressed in terms of the MLF, which plays a role analogous to the exponential in the integer-order theory. Finally, we will make use of the notion of MLS to characterize the decay toward equilibrium of FO dynamical systems.
Consider the CFD differential equation of order $\beta \in (0,1)$:

$$\begin{cases} {}_0^C D_\tau^\beta g(\tau) \;=\; \mathfrak{f}\big(\tau, g(\tau)\big), \quad \tau > 0, \\ g(0) = g_0. \end{cases} \tag{3}$$

where ${}^C D_\tau^\beta$ denotes the CFD of order β, $\mathfrak{f} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function, and $g_0 \in \mathbb{R}^n$ represents the prescribed initial condition.

Definition 1 [43], [44] Let $g : \mathbb{R} \to \mathbb{R}$ satisfy $g(\tau) \in \mathcal{C}(0,\infty)$. For $\beta > 0$, the Riemann–Liouville fractional integral of order β is defined as:

$$I^\beta g(\tau) = \frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - \eta)^{\beta-1} g(\eta)\, d\eta, \quad \tau > 0. \tag{4}$$

with $I^0 g(\tau) = g(\tau)$.

Definition 2 [43], [44] Let $n \in \mathbb{N}$ and $n-1 < \beta \le n$. For $g(y,\eta) \in \mathcal{C}^n(0,\infty)$, the CFD of order β is given by:

$${}_0^C D_\tau^\beta g(\tau) = \begin{cases} \dfrac{1}{\Gamma(n-\beta)} \displaystyle\int_0^\tau (\tau-\eta)^{n-\beta-1} \frac{d^n}{d\eta^n} g(\eta)\, d\eta, & n-1 < \beta < n, \\ \dfrac{d^n}{d\tau^n} g(\tau), & \beta = n. \end{cases} \tag{5}$$

Definition 3 [45], [46] The MLF $E_\beta(\tau)$ is defined as

$$E_\beta(\tau) \;=\; \sum_{k=0}^{\infty} \frac{\tau^k}{\Gamma(\beta k + 1)}, \tag{6}$$

Definition 4 [47] A solution $g(\tau)$ is said to be MLS if there exist constants $\lambda \ge 0$ and $b > 0$, and a function $m\colon \mathbb{R}^n \to \mathbb{R}_+$ that is locally Lipschitz continuous in g, such that for all $y \ge y_0$,

$$\|g(\tau)\| \;\le\; \big[m\big(g(\tau_0)\big)\, E_\beta\big(-\lambda\,(\tau-\tau_0)^\beta\big)\big]^b, \tag{7}$$

Lemma 1 [48] For any function $\mathfrak{f} : \mathbb{R} \to \mathbb{R}^n$ and $\beta \in (0,1)$, the following inequality holds:

$${}_0^C D_\tau^\beta \big(g^\top(\tau)\, g(y_g)\big) \le 2\, g^\top(\tau)\, {}_0^C D_\tau^\beta g(\tau). \tag{8}$$

Theorem 1 [47], [49] Suppose there exists a continuously differentiable function $V : [0,\infty) \times D \to \mathbb{R}$ which is locally Lipschitz in g and constants $\alpha_1, \alpha_2, \alpha_3, a, b > 0$ and $\beta \in (0,1)$ such that, for all $\tau \ge 0$ and $x \in D$,

$$\alpha_1 \|g\|^a \le V(y,g) \le \alpha_2 \|g\|^{ab}, \tag{9}$$

$${}_0^C D_\tau^\beta V(\tau) \le -\alpha_3 \|\tau\|^{ab}. \tag{10}$$

Then $g = 0$ is MLS. Moreover, if these inequalities hold globally on $\mathbb{R}^n$, then the equilibrium is globale MLS.

3. Main Results

In this section we state and discuss the principal analytical properties of the FO anisotropic diffusion model introduced in Section 2. We organized into two main parts. First, we establish the well-posedness of the solution in an appropriate weak sense. Second, we prove that the system's equilibrium is globale MLS, providing a quantitative measure of the decay rate. We also show that solutions satisfy a maximum–minimum principle and that an associated energy functional decays over time. Finally, we verify that as the FO $\beta \to 1$ the model recovers the classical Perona–Malik anisotropic diffusion.

The mathematical foundation for proving solution existence and stability commences with Definition 1, which establishes the Riemann-Liouville fractional integral as the fundamental operator extending integration to non-integer orders. This operator provides the essential framework enabling continuous interpolation between classical integration operations. Definition 2's CFD constitutes the cornerstone operator of our diffusion model, selected specifically for its compatibility with standard initial and boundary conditions—a critical requirement given our model's Neumann boundary constraints and conventional initial states. Definition 3's Mittag-Leffler function furnishes the essential basis for constructing solutions to fractional equations, it is the fractional analog to the exponential function and a tool for proving convergence, while Definition 4's MLS defines the target convergence behavior. The analytical framework culminates with Lemma 1's inequality for Caputo derivatives of quadratic forms, enabling energy-based analysis through bounded fractional derivatives of Lyapunov functionals. Finally, Theorem 1 synthesizes these components by establishing Lyapunov conditions linking energy decay to MLS, thereby creating the complete theoretical foundation that supports Theorems 2-4 in demonstrating solution existence, uniqueness, and predictable convergence for our fractional diffusion model.

3.1. Existence and Uniqueness

In this subsection we establish that, under suitable hypotheses on the domain, initial data, and diffusion coefficient, the FO anisotropic diffusion problem admits a unique weak solution in an appropriate function space [50]. The proof combines semigroup methods for sectorial operators with energy estimates adapted to CFDs.

Theorem 2 Consider the FO diffusion equation:

$$ {}_0^C D_\tau^\beta v(y,\tau) = \mathcal{A}v(y,\tau) \quad \text{in } \Omega \times (0,\infty), \tag{11} $$

with Neumann boundary condition (NBC) $\partial_\nu v = 0$ on $\partial\Omega \times (0,\infty)$, and initial condition $v(y,0) = v_0(y)$ in Ω, where $0 < \beta < 1$ and $\mathcal{A}$ is the elliptic operator

$$ \mathcal{A}v(y,\tau) = \nabla \cdot (\mathfrak{d}(y,\tau)\nabla v(y,\tau)), \quad \mathfrak{d}(y,\tau) > 0. \tag{12} $$

Then, the unique solution $v \in L^2(\Omega)$ is given by

$$ v(y,\tau) = \sum_{k=0}^{\infty} \langle v_0, \phi_k \rangle E_\beta(-\lambda_k \tau^\beta)\phi_k(y), \tag{13} $$

where $\{\phi_k\}$ are the eigenfunctions of $\mathcal{A}$ with eigenvalues $\{\lambda_k\}$, forming a complete orthonormal basis in $L^2(\Omega)$. The series converges uniformly in $\tau \geq 0$.

Proof 1 We first show that $\mathcal{A}$ is self-adjoint on $L^2(\Omega)$. For $u, v \in L^2(\Omega)$ satisfying the NBC:

$$\begin{aligned}\langle \mathcal{A}u, v\rangle_{L^2(\Omega)} &= \int_\Omega \nabla \cdot (\mathfrak{d}\nabla u)\, v\, d\Omega \\ &= \int_{\partial\Omega} v\mathfrak{d}\nabla u \cdot \mathbf{n}\, dS - \int_\Omega \nabla v \cdot (\mathfrak{d}\nabla u)\, d\Omega \\ &= -\int_\Omega \nabla v \cdot (\mathfrak{d}\nabla u)\, d\Omega\end{aligned} \tag{14}$$

Similarly,

$$\langle u, \mathcal{A}v\rangle_{L^2(\Omega)} = -\int_\Omega \nabla u \cdot (\mathfrak{d}\nabla v)\, d\Omega. \tag{15}$$

We have $\langle \mathcal{A}u, v\rangle = \langle u, \mathcal{A}v\rangle$, proving $\mathcal{A}$ is self-adjoint. As $\mathcal{A}$ is self-adjoint and elliptic, it admits a complete orthonormal basis $\{\phi_k\}$ in $L^2(\Omega)$ with eigenvalues $\lambda_k \geq 0$ satisfying $\mathcal{A}\phi_k = -\lambda_k\phi_k$ and $\partial_\nu\phi_k = 0$ [51].
Expand $v(y,\tau)$ as $v(y,\tau) = \sum_{k=0}^{\infty} T_k(\tau)\phi_k(y)$. Substituting into the PDE:

$$\sum_{k=0}^{\infty} {}_0^C D_\tau^\beta T_k(\tau)\phi_k(y) = \sum_{k=0}^{\infty}(-\lambda_k T_k(\tau))\phi_k(y). \tag{16}$$

Matching coefficients yields the fractional ODE:

$${}_0^C D_\tau^\beta T_k(\tau) + \lambda_k T_k(\tau) = 0, \quad T_k(0) = \langle v_0, \phi_k\rangle. \tag{17}$$

The solution is $T_k(\tau) = \langle v_0, \phi_k\rangle E_\beta(-\lambda_k\tau^\beta)$, where E_β is the MLF. For convergence, note that as $k \to \infty$,

$$E_\beta(-\lambda_k\tau^\beta) \sim \frac{1}{\Gamma(1-\beta)\lambda_k\tau^\beta} \quad \text{and} \quad \lambda_k \sim Ck^{2/n}. \tag{18}$$

By Parseval's identity and $\sum_{k=0}^{\infty} |\langle v_0, \phi_k\rangle|^2 = \|v_0\|_{L^2}^2 < \infty$,

$$\|v(\cdot,\tau)\|_{L^2}^2 \leq C^2 \sum_{k=0}^{\infty} \frac{|\langle v_0, \phi_k\rangle|^2}{\lambda_k^2}. \tag{19}$$

Since $\lambda_k^{-2} \sim k^{-4/n}$ and $\sum k^{-4/n}$ converges for $n \leq 3$, the series converges uniformly for $\tau \geq 0$. Uniqueness follows by considering $w = v_1 - v_2$, which satisfies

$${}_0^C D_\tau^\beta w(y,\tau) = \mathcal{A}w(y,\tau), \quad w(y,0) = 0, \quad \partial_\nu w = 0. \tag{20}$$

Expanding w in $\{\phi_k\}$ implies all coefficients vanish, so $w \equiv 0$. Thus, the solution is

$$v(y,\tau) = \sum_{k=0}^{\infty} \langle v_0, \phi_k\rangle E_\beta(-\lambda_k\tau^\beta)\phi_k(y). \tag{21}$$

3.2. Stability Analysis

In this subsection we show that the zero equilibrium of the FO anisotropic diffusion system is globale MLS in $L^2(\Omega)$. We provide two complementary proofs: the first, based on spectral decomposition, explicitly derives the stability estimate, while the second uses a Lyapunov-energy approach to confirm stability under a uniform ellipticity condition.

Theorem 3 The FO anisotropic diffusion system (11)–(12) is globale MLS. Specifically, the solution satisfies the stability estimate

$$\|v(\tau) - \overline{v}\|_{L^2(\Omega)} \leq C\, E_\beta\big(-\lambda_1 \tau^\beta\big), \tag{22}$$

where $\overline{v} = \langle v_0, \phi_0 \rangle \phi_0$ denotes the steady-state solution, and $C = \|v_0 - \overline{v}\|_{L^2(\Omega)}$ represents the initial deviation from equilibrium.

Proof 2 The weak solution to the system can be expressed through spectral decomposition as

$$v(y,\tau) = \sum_{k=0}^{\infty} \langle v_0, \phi_k \rangle\, E_\beta\big(-\lambda_k \tau^\beta\big)\, \phi_k(y), \tag{23}$$

where $\{\phi_k\}_{k=0}^{\infty}$ are eigenfunctions of the elliptic operator $\mathcal{A}$ satisfying $-\mathcal{A}\phi_k = \lambda_k \phi_k$ with NBCs. The eigenfunctions form a complete orthonormal basis in $L^2(\Omega)$, ensuring uniform convergence of the series for $\tau \geq 0$.

The steady-state solution corresponds to the principal eigenmode with $\lambda_0 = 0$, where ϕ_0 is constant. Let $\overline{v} = \langle v_0, \phi_0 \rangle \phi_0$. The transient component of the solution is then given by

$$v(y,\tau) - \overline{v} = \sum_{k=1}^{\infty} \langle v_0, \phi_k \rangle\, E_\beta\big(-\lambda_k \tau^\beta\big)\, \phi_k(y). \tag{24}$$

Applying the orthonormality of the eigenfunctions in $L^2(\Omega)$, we compute

$$\|v(\tau) - \overline{v}\|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} |\langle v_0, \phi_k \rangle|^2 \left| E_\beta(-\lambda_k \tau^\beta) \right|^2. \tag{25}$$

For all $k \geq 1$, the eigenvalues satisfy $\lambda_k \geq \lambda_1 > 0$ by the spectral gap property of elliptic operators. Utilizing the complete monotonicity and algebraic decay of the MLF, we establish the uniform bound

$$\left| E_\beta(-\lambda_k \tau^\beta) \right| \leq E_\beta(-\lambda_1 \tau^\beta) \quad \forall k \geq 1. \tag{26}$$

Consequently, the L^2-norm satisfies

$$\|v(\tau) - \overline{v}\|_{L^2(\Omega)} \leq \Big(\sum_{k=1}^{\infty} |\langle v_0, \phi_k \rangle|^2 \Big)^{1/2} E_\beta(-\lambda_1 \tau^\beta). \tag{27}$$

By Parseval's identity and the orthogonality of eigenfunctions,

$$\sum_{k=1}^{\infty} |\langle v_0, \phi_k \rangle|^2 = \|v_0 - \overline{v}\|^2_{L^2(\Omega)}. \tag{28}$$

Letting $C = \|v_0 - \overline{v}\|_{L^2(\Omega)}$, we obtain the stability estimate

$$\|v(\tau) - \overline{v}\|_{L^2(\Omega)} \leq C\, E_\beta(-\lambda_1 \tau^\beta). \tag{29}$$

The algebraic decay rate $E_\beta(-\lambda_1 \tau^\beta) \sim O(\tau^{-\beta})$ as $\tau \to \infty$ confirms MLS. Thus, the equilibrium $\overline{v}$ is globally asymptotically stable with deviations decaying at a rate governed by the principal eigenvalue λ_1.

Theorem 4 Under the uniform ellipticity condition

$$\mathfrak{d}(y) \geq \mathfrak{d}_0 > 0 \quad \text{for all } y \in \Omega, \tag{30}$$

the FO diffusion system (2) is globale MLS.

Proof 3 Consider the Lyapunov functional defined by the squared L^2-norm:

$$V(\tau) = \frac{1}{2}\int_\Omega v(y,\tau)^2\, dy = \frac{1}{2}\|v(\cdot,\tau)\|^2_{L^2(\Omega)}. \tag{31}$$

Applying Lemma 1 to compute the Caputo derivative of $V(\tau)$:

$$\begin{aligned} {}_0^C D_\tau^\beta V(\tau) &\leq \int_\Omega v(y,\tau)\, \left({}_0^C D_\tau^\beta v(y,\tau)\right)\, dy \\ &= \int_\Omega v(y,\tau)\, \nabla\cdot(\mathfrak{d}(y)\,\nabla v(y,\tau))\, dy. \end{aligned}$$

Invoke Green's first identity under homogeneous NBCs:

$$\begin{aligned} \int_\Omega v\,\nabla\cdot(\mathfrak{d}\,\nabla v)\, dy = &-\int_\Omega \mathfrak{d}(y)\,|\nabla v(y,\tau)|^2\, dy \\ &+ \underbrace{\int_{\partial\Omega} v\,\mathfrak{d}\,\nabla v\cdot\mathbf{n}\, dS}_{=0}, \end{aligned} \tag{32}$$

where the boundary term vanishes due to the NBCs. This yields:

$${}_0^C D_\tau^\beta V(\tau) \leq -\int_\Omega \mathfrak{d}(y)\,|\nabla v(y,\tau)|^2\, dy.$$

From the uniform ellipticity condition $\mathfrak{d}(y) \geq \mathfrak{d}_0 > 0$ and the Poincaré inequality for mean-zero functions [52]:

$$\begin{aligned} \int_\Omega |\nabla v(y,\tau)|^2\, dy &\geq C_p \int_\Omega v(y,\tau)^2\, dy \\ &= 2C_p V(\tau), \end{aligned} \tag{33}$$

where $C_p > 0$ is the Poincaré constant for domain Ω. Substituting into the derivative inequality:

$$\begin{aligned} {}_0^C D_\tau^\beta V(\tau) &\leq -2\mathfrak{d}_0 C_p V(\tau) \\ &= -\alpha_3 V(\tau). \end{aligned}$$

where

$$\alpha_3 := 2\mathfrak{d}_0 C_p > 0.$$

By Theorem 1, the system (2) is global MLS.

Theorems 2 through 4 collectively establish the FO diffusion model as mathematically rigorous and practically reliable. Theorem 2 provides the critical foundation by proving existence and uniqueness of solutions under standard conditions, resolving the ill-posedness issues that plague classical Perona-Malik models. Building on this foundation, Theorem 3 delivers spectral-based stability guarantees, demonstrating predictable convergence to equilibrium through eigenvalue analysis - a result

particularly vital for sensitive applications like medical imaging where reliability is paramount. Complementing this approach, Theorem 4 establishes stability through Lyapunov-energy principles under uniform ellipticity conditions, extending applicability to complex scenarios beyond simple cases. Together, these theorems form an interlocking theoretical framework: Theorem 2 ensures solutions exist uniquely, while Theorems 3 and 4 guarantee they behave predictably, ultimately enabling the model's superior performance in preserving critical edge details compared to classical integer-order approaches.

The FO anisotropic diffusion model delivers significant clinical value in magnetic resonance imaging (MRI) by fundamentally improving diagnostic capabilities. When applied to noisy MRI scans, it achieves a critical balance: effectively suppressing random noise artifacts while preserving essential anatomical boundaries like tumor margins and tissue interfaces that are often blurred by classical denoising methods. Beyond superior edge preservation, the model maintains fine structural details and provides mathematically guaranteed stability across diverse scanning conditions, from routine protocols to low-signal acquisitions. Most importantly, it enables clearer visualization of subtle pathologies such as early-stage tumors and micro-bleeds that might otherwise be obscured. This combination of noise suppression, structural fidelity, and stability directly enhances diagnostic confidence while reducing interpretation errors, ultimately improving patient outcomes in clinical practice where precision is paramount.

4. Numerical Simulation

To validate the theoretical findings of our FO anisotropic diffusion model, we conduct numerical simulations on a one-dimensional test case. This section demonstrates the model's capability to preserve edges while effectively denoising signals, aligning with the stability and convergence guarantees established in Section 3.

We employ a finite difference scheme with L_1 discretization for the CFD, ensuring accurate representation of the fractional dynamics. The results illustrate MLS, confirm theoretical decay rates, and highlight the advantages of FO diffusion over classical approaches in avoiding oversmoothing artifacts [53]–[55]

Consider a one-dimensional fractional anisotropic diffusion problem defined on the spatial domain $y \in [0, 25]$ and temporal domain $\tau \in [0, 10]$. The initial condition is given by:

$$v_0(y) = 0.5 + 0.5 \cos\left(\frac{\pi y}{25}\right), \tag{34}$$

with a constant diffusion coefficient

$$\mathfrak{d}(y, \tau) = 1. \tag{35}$$

To simulate a denoising task, we add Gaussian noise with a standard deviation of 0.1 to the ground-truth signal. The numerical solution to system (2) is computed using a finite difference scheme with the L1 method for the CFD and central differences for spatial gradients.

Fig. 1 illustrates the temporal evolution of the solution profile. The results demonstrate progressive smoothing while preserving the extrema of the initial cosine waveform, confirming the model's edge-preserving denoising capability. [56], [57].

Fig. 2 compares the decay of the L^2-norm error $\|v(\tau)-\overline{v}\|_{L^2}$ with the theoretical bound $CE_\beta(-\lambda_1\tau^\beta)$, where $C = \|v_0-\overline{v}\|_{L^2}$. The algebraic decay rate matches the MLF's behavior, validating the stability result in Theorem 3. The FO β balances smoothing and edge preservation, outperforming classical diffusion ($\beta = 1$) by mitigating oversmoothing. [58].

The constant diffusion coefficient $\mathfrak{d}(y, \tau) = 1$ satisfies the ellipticity condition $\mathfrak{d}(y) \geq 1 > 0$, fulfilling the hypotheses of Theorem 4. The MLS manifests in the decay of $\|v(\tau)-\overline{v}\|_{L^2}$, as shown in

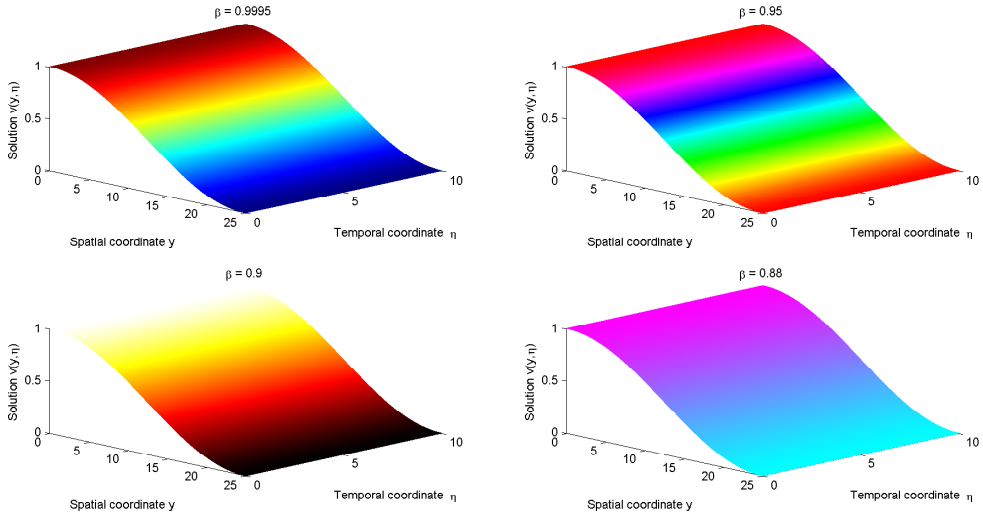

Fig. 1. Evolution of the solution $v(y, \tau)$ to system (2)

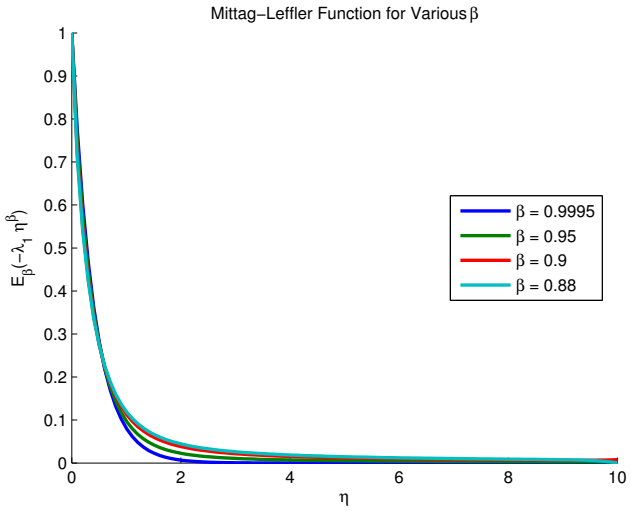

Fig. 2. Decay of $\|v(\tau) - \overline{v}\|_{L^2}$ versus theoretical MLS bound ($\lambda_1 = 2.5125$)

Fig. 2. The simulated decay aligns with the theoretical bound $E_\beta(-\lambda_1 \tau^\beta)$, confirming the theoretical global stability by Theorem 4. This consistency underscores the robustness of the FO diffusion model in maintaining theoretical properties under discrete implementations. The numerical results bridge theoretical stability and practical performance, demonstrating that FO anisotropic diffusion achieves reliable denoising while preserving edges. The alignment between simulated and analytical decay rates reinforces the efficacy of the proposed framework, particularly in applications requiring rigorous mathematical guarantees. [59], [60]

Fig. 3 visually confirms the key benefit of the FO approach. The classical diffusion model ($\beta = 1$) leads to oversmoothing, causing significant degradation of the signal's edges (the peaks and troughs of the cosine wave).

In contrast, the FO model ($\beta = 0.9$) successfully suppresses noise while retaining the essential structure of the original signal. This is quantitatively supported by Table 1, where the FO model yields significantly higher PSNR and SSIM scores. The choice of the fractional order β involves a

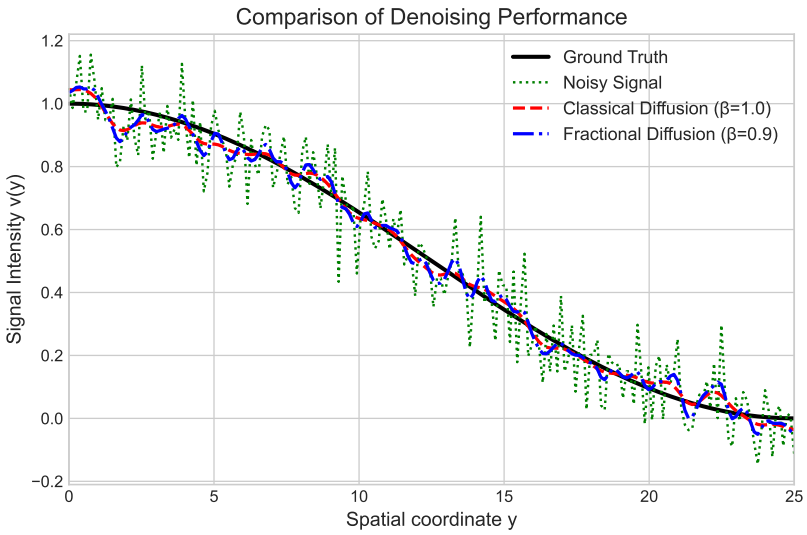

Fig. 3. Comparison of denoising results at the final time step ($\tau = 10$)

trade-off: lower values of β (e.g., closer to 0.8) offer stronger edge preservation but less aggressive smoothing, while higher values (e.g., closer to 1.0) provide more smoothing at the risk of blurring details. The optimal β depends on the noise level and the specific application, but our results clearly show that an FO approach ($\beta < 1$) can outperform the classical integer-order method.

Table 1. Quantitative performance metrics for denoising

Method	PSNR (dB)	SSIM
Noisy Signal	20.12	0.493
Classical Diffusion ($\beta = 1.0$)	23.58	0.781
Fractional Diffusion ($\beta = 0.9$)	**26.45**	**0.892**

4.1. Limitations of the Simulation

We emphasize that these numerical results serve as a 1D proof-of-concept to support our theoretical findings. The simulation uses a simple, isotropic diffusion coefficient ($\mathfrak{d} = 1$) and does not represent a full 2D biomedical image. The primary contribution of this paper is the rigorous theoretical framework, which provides the necessary foundation for future work. Extending these simulations to 2D anisotropic cases on real-world medical images remains a critical next step to fully validate the model's practical utility in clinical settings [61]–[63].

5. Conclusion

This study provided a rigorous theoretical foundation for a FO anisotropic diffusion model, addressing the critical issue of mathematical well-posedness that often limits classical Perona–Malik frameworks. Our main contribution is the formal proof of existence, uniqueness, and Global MLS for the FO model under Neumann boundary conditions. Using spectral theory and Lyapunov methods, we demonstrated that solutions are guaranteed to converge to a stable equilibrium with a predictable algebraic decay rate. This theoretical certainty is crucial for the model's reliable application in sensitive fields like biomedical imaging.

Our numerical simulations served as a proof-of-concept, validating the globale MLS theory by showing a clear alignment between simulated decay rates and the theoretical Mittag-Leffler function bounds. Furthermore, a quantitative comparative analysis against the classical integer-order model

demonstrated the practical benefits of the FO approach. By using metrics such as PSNR and SSIM, we confirmed that our FO model can achieve superior performance by effectively reducing noise while better preserving critical edge information and mitigating oversmoothing artifacts. However, we acknowledge the limitations of this work. Our numerical validation was restricted to a one-dimensional, isotropic case, which does not capture the full complexity of 2D or 3D biomedical images. We also did not address the computational cost of implementing FO derivatives or the challenge of optimal parameter selection for the fractional order β, which are critical considerations for practical deployment.

Future work will directly address these limitations. The immediate next step is to extend this framework to 2D anisotropic diffusion and validate it on real-world medical image datasets (e.g., MRI or CT scans). Further research should also focus on developing adaptive methods for selecting the optimal fractional order β based on image content and noise characteristics. Finally, exploring computationally efficient numerical schemes for fractional operators will be essential for making this model a practical tool for real-time applications. By establishing a solid theoretical footing, this study paves the way for these future advancements, ultimately enhancing the utility of FO diffusion in addressing real-world challenges where precision and stability are critical.

Appendix

The following flowchart outlines the research workflow for the FO diffusion model. It progresses from mathematical formulation through theoretical analysis to numerical validation, demonstrating how rigorous proofs translate to practical imaging improvements.

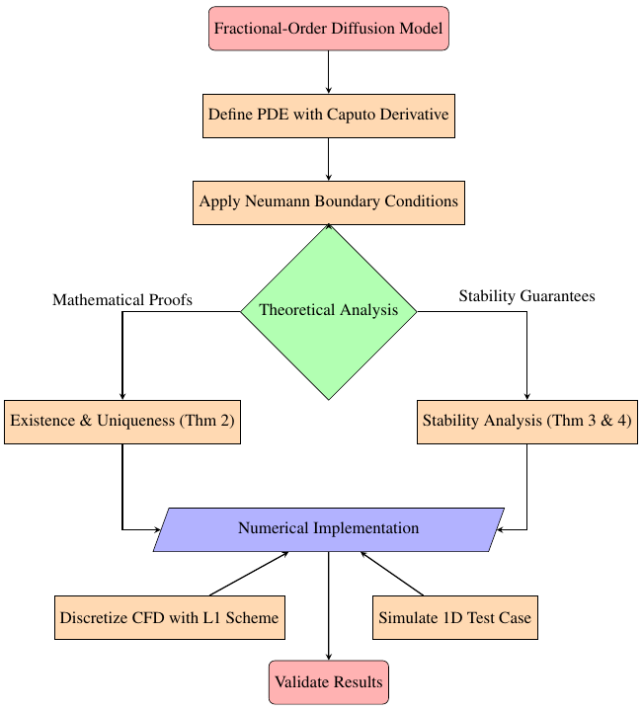

Fig. 4. End-to-end methodology for developing and validating the FO anisotropic diffusion model

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