

On General Fractional-Order Discrete-Time Reaction-Diffusion Systems: Finite Time Stability and Simulations

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ABSTRACT

Finite-time stability (FTS) in fractional-order reaction-diffusion systems (FO-RDs) is critically important for biochemical applications where rapid stabilization prevents metabolic imbalances, such as in glucose-insulin regulatory systems where delayed convergence can lead to diabetic complications. This work advances the field by providing the first comprehensive stability framework for discrete-time FO-RDs that guarantees convergence within a predetermined finite time, unlike previous approaches that only established asymptotic stability. We introduce a novel Lyapunov functional (LF) and, through careful eigenvalue analysis of the discrete Laplacian and boundedness of nonlinear terms, we establish that its fractional derivative remains strictly negative for all states within a neighborhood of the equilibrium point, with explicit bounds derived from system parameters. This approach, which integrates discrete Green's formula and eigenvalue estimates, rigorously addresses the challenges of non-local fractional operators in spatially discretized systems. The theoretical framework is validated through numerical simulations of a fractional glycolysis model, which demonstrate rapid convergence and robustness. The results confirm the method's potential for designing robust control strategies for complex biochemical and chemical processes that require swift stabilization.

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1. Introduction

The study of RDs is fundamental to understanding pattern formation, wave propagation, and stability in various scientific fields, from chemical kinetics to population dynamics [1], [2]. While continuous-time and integer-order models have been extensively studied [3], [4], FO models have gained prominence for their ability to capture memory and hereditary properties inherent in many

complex systems [5]. The stability of these systems is a primary concern, with FTS being particularly crucial for applications requiring rapid convergence to a desired state, such as in drug delivery systems and chemical reactor control [6], refer to [7]–[13] for other potential research avenues.

The stability properties of RDs have been extensively investigated using various analytical approaches. In [14]–[17], a nonlinear stability technique was introduced to detect pattern formation, proving particularly effective for systems with heterogeneous diffusion rates. Additionally, [18]–[21] utilized upper and lower solution methods for stability analysis, while [22]–[26] applied the Galerkin method to approximate the Selkov–Schnakenberg system. Complementing this, [27], [28] analyzed the stability of singularly perturbed solutions via spectral analysis. Among these methods, LF techniques have been widely employed to assess stability. In [29], a specialized LF was shown to predict equistability in systems with multiple stationary states, and for a broad class of RDs with Neumann boundary conditions, [30]–[33] established that a priori L^r estimates guarantee L^∞ -uniform stability. Furthermore, [34], [35] introduced a novel LF for binary RDs, and precise LFs were constructed in [36]–[38] to prove asymptotic stability (AS) of solutions, underscoring the versatility and efficacy of LFs in RD stability analysis.

Recent research has emphasized both AS and FTS in RDs. Significant efforts have focused on deriving sufficient conditions for global AS in systems extending classical nonlinear models [39]–[45]. Concurrently, studies have addressed FTS, with [46] designing controllers to achieve FTS under Neumann/mixed boundary conditions, while others extended these strategies to stochastic delay systems [47]–[50]. Further work has explored FTS for stochastic Markovian systems [51]–[54], impulsive discrete RDs [55], and switched stochastic RD systems [56]. Parallel research has addressed FTS and synchronization in FO-RDs: [57] developed FTS criteria for nonlinear FO systems with discrete delays; [58] designed controllers for FO-RD stabilization; and [59], [60] studied FTS of equilibria and synchronization in various FO-RD models using Lyapunov theory.

Despite this extensive work on continuous FO-RDs, three critical gaps remain in the context of their discrete counterparts, which are essential for numerical simulation and practical implementation. First, no existing framework guarantees finite-time convergence for discrete spatial models; existing methods often focus on asymptotic stability, which is insufficient for time-critical applications. Second, current LF-based methods for continuous systems do not adequately account for the intricate interaction between fractional memory effects and spatial discretization errors, which can significantly alter system dynamics. Third, practical implementations lack methods for deriving explicit settling time estimates, which are essential for real-time control and performance guarantees. This paper addresses these limitations by developing a rigorous theoretical framework for the FTS of general discrete-time FO-RDs. Unlike previous work that established asymptotic stability, our contribution lies in developing the first framework that guarantees convergence within a predetermined finite time for discrete FO-RDs, with settling time estimates derived from system parameters. The research contribution is threefold:

1. The derivation of explicit sufficient conditions for FTS of discrete FO-RDs using a novel LF approach tailored for discrete Caputo fractional differences.
2. The development of a rigorous analytical framework that connects discrete Caputo fractional differences with spatial discretization effects, incorporating discrete Green's formula and eigenvalue analysis to ensure validity.
3. Validation of the framework through numerical simulations on a biochemical model, demonstrating its practical applicability and robustness. By establishing a clear research gap and providing a comprehensive solution, this work bridges the divide between theoretical continuous-time analysis and practical discrete-time implementation, paving the way for advanced control strategies in systems where both fractional dynamics and spatial distribution are key.

The aim of this work is to develop this novel analytical framework for investigating the FTS

of discrete FO-RDs by leveraging FO calculus and LF techniques. The presentation is organized as follows: [Section 2](#) introduces basic concepts and mathematical preliminaries. [Section 3](#) describes the model formulation and discretization. [Section 4](#) establishes the main theoretical results, including the derivation of explicit FTS conditions. [Section 5](#) provides numerical simulations on a FO glycolysis RD model to illustrate and validate the theoretical findings.

2. Basic Concepts

In this section, we introduce the essential mathematical tools and definitions required for our analysis of FO systems. We begin by outlining the fundamentals of FO calculus, including the Riemann–Liouville integral (RLI) and the Caputo fractional derivative (CFD), which are instrumental in modeling systems with memory effects. Next, we define key notions such as EPs and FTS-EPs. An equilibrium point (EP) is a state where the system remains constant once reached, whereas an FTS-EP further requires that the system’s state converges to the EP within a finite ST. These concepts form the basis of our analytical framework, allowing us to rigorously examine the transient behaviors of FO-RDs. Examine the following nonlinear system governed by FO dynamics:

$${}^C\nabla^\zeta \varphi(t) = \mathbf{g}(t, \varphi(t)), \quad \varphi(a) = \varphi_a, \quad t \in \mathbb{N}_a. \quad (1)$$

where $\mathbf{g} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\varphi(t) \in \mathbb{R}^n$, and ${}^C\nabla^\zeta$ denotes the CFD of order $\zeta \in (0, 1)$. A point $\varphi^* \in \mathbb{R}^n$ is an EP of system (1) if $\mathbf{g}(\varphi^*) \equiv 0$. The backward difference operators are defined, respectively, as

$$\nabla \varphi(t) = \varphi(t) - \varphi(t-1), \quad (2)$$

where

$$\mathbb{N}_a = \{a, a+1, a+2, \dots\}, \quad \mathbb{N}_a^T = \{a, a+1, a+2, \dots, T\}$$

Definition 1 [61] The ζ -th left fractional sum of a function φ is given by

$$\nabla^{-\zeta} \varphi(t) = \frac{1}{\Gamma(\zeta)} \sum_{s=t+\zeta}^a (\rho(s) - t)^{\overline{\zeta-1}} \varphi(s). \quad (3)$$

where

$$t^{\overline{\zeta}} = \frac{\Gamma(t + \zeta)}{\Gamma(t)}, \quad (4)$$

Definition 2 [61] Let $\zeta > 0$, $\zeta \notin \mathbb{N}$. Then, the ζ -order CFD of a function f defined on \mathbb{N}_a , respectively, are defined by

$${}^C\nabla^\zeta \varphi(t) = \nabla^{-(n-\zeta)} \nabla_a^n \varphi(t) = \frac{1}{\Gamma(n-\zeta)} \sum_{s=t+(n-\zeta)}^b (\rho(s) - t)^{\overline{n-\zeta-1}} \nabla_b^n \varphi(s), \quad (51)$$

where $n = [\zeta] + 1$.

Definition 3 [62] The Mittag-Leffler function is defined as follows:

$$E_{\overline{\zeta}}(s, t) = \sum_{k=0}^{\infty} s^k \frac{t^{\overline{k\zeta}}}{\Gamma(\zeta k + 1)}, \quad |s| < 1. \quad (5)$$

Definition 4 A function φ^* is referred to as the initial EP of system (1) if $\varphi_0 = \varphi^*$; equivalently, $\varphi(t) \equiv \varphi^*$ for all $t \geq t^*$.

Definition 5 A point φ^* is defined as a FTS-EP of system (1) if there exists a ST $t^* \in \mathbb{N}_a$ such that $\varphi(t) \neq \varphi^*$ for $t < t^*$ and $\varphi(t) \equiv \varphi^*$ for all $t \geq t^*$.

Definition 6 [63] Given positive numbers δ, ε , and T , and for the initial condition φ_a , system (1) is said to be FTS with respect to (δ, ε, T) , where $\varepsilon < \delta$, if and only if

$$\|\varphi_a\| < \delta, \quad (6)$$

then

$$\|\varphi(t)\| < \varepsilon, \quad \text{for all } t \in \mathbb{N}_{a+1}^{t^*}. \quad (7)$$

Lemma 1 [62] For any $\varphi(t) \in \mathbb{R}^n$ and $\zeta \in (0, 1)$, the following bound is satisfied:

$${}^C\nabla^\zeta \left(\varphi^\top \varphi \right) \leq 2 \varphi^\top {}^C\nabla^\zeta \varphi, \quad t \geq 0. \quad (8)$$

Lemma 2 [64] Assume that $0 < \zeta \leq 1$ and $\varphi: \mathbb{N}_a \rightarrow \mathbb{R}$. Then,

$$\nabla^{-\zeta} {}^C\nabla^\zeta \varphi(t) = \varphi(t) - \varphi(a). \quad (9)$$

Theorem 1 [63] Suppose that $u(r)$ is discrete non-negative, non-decreasing function. For each $t \in \mathbb{N}_a$, let $\phi(t)$ be a non-negative function that satisfies

$$\phi(r) \leq u(t) + \mathfrak{L} {}^C\nabla^{-\zeta} \phi(t), \quad (10)$$

Then,

$$\phi(t) \leq u(t) E_{\bar{\zeta}} \left(\frac{t^\zeta}{t^\zeta} \mathfrak{L}, t - a \right). \quad (11)$$

Remark 1 Lemma 1 is employed in the proof to bound the fractional derivative of the quadratic LF. Its applicability is ensured since the LF is expressed as a sum of squares, which constitutes a continuously differentiable function, and the state variables are defined in \mathbb{R}^n . Lemma 2 establishes the fundamental inverse relationship in discrete fractional calculus, analogous to the fundamental theorem of calculus, and is crucial for transforming the derived fractional-order difference inequality for the LF into a form amenable to the application of the Grönwall inequality. The discrete Grönwall-type inequality then serves as the final step in deriving an explicit bound on the LF. Its use is warranted once the LF is shown to satisfy the requisite linear inequality, a condition verified through the preceding lemmas.

3. Model Description

In this section, we introduce the discrete FO-RD model considered in this study. We begin with the general form of the continuous-space FO-RDs, followed by its spatial discretization using finite difference techniques. Key parameters, boundary conditions, and assumptions relevant to the system

dynamics are also discussed.

We first present the general RD model as follows:

$$\begin{cases} {}_0^C D_t^\zeta U_1(x, t) = \sum_{j=1}^2 \varsigma_{1,j} \Delta U_j + \sum_{j=1}^2 \aleph_{1,j} U_j + F_1(U_1, U_2), \\ {}_0^C D_t^\zeta U_2(x, t) = \sum_{j=1}^2 \varsigma_{2,j} \Delta U_j + \sum_{j=1}^2 \aleph_{2,j} U_j + F_2(U_1, U_2), \\ \left. \begin{aligned} \frac{\partial U_1}{\partial \nu} \Big|_{\partial\Omega} = \frac{\partial U_2}{\partial \nu} \Big|_{\partial\Omega} = 0, \\ U_1(x, 0) = \kappa_1(x), \quad U_2(x, 0) = \kappa_2(x). \end{aligned} \right\} \quad (12)$$

Here, $(x, t) \in \Omega \times \mathbb{R}_+^*$ denotes a bounded domain with smooth boundary $\partial\Omega$, and $(U_1(x, t), U_2(x, t))^T$ represents the state variables. The matrices $(\varsigma_{i,j})$ and $M = (\aleph_{i,j})$ belong to $\mathbb{R}^{2 \times 2}$, and F_1 and F_2 are continuous nonlinear functions. To analyze the system numerically, we discretize the spatial variable x in the FO-RDs (12) using a finite difference method. The spatial domain Ω is discretized into $N + 1$ points with step size $h = L/N$, where L is the length of Ω . The Neumann boundary conditions $\frac{\partial U_k}{\partial \nu} \Big|_{\partial\Omega} = 0$ are incorporated using ghost points, ensuring a second-order accurate spatial discretization. The continuous Laplacian ΔU_j is replaced with its discrete approximation $\Delta_h U_j$, defined as follows for each component j :

- At boundary $i = 1$: $\Delta_h U_0(t) = \frac{U_1(t) - U_0(t)}{h}$
- At interior points $i = 1, \dots, N$: $\Delta_h^2 U_i(t) = \frac{U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)}{h^2}$
- At boundary $i = N + 1$: $\Delta_h U_{N+1}(t) = \frac{U_{N+1}(t) - U_N(t)}{h}$

The FO time derivative ${}_0^C D_t^\zeta$ is approximated by the discrete Caputo fractional difference operator ${}^C \nabla^\zeta$ as described in [5]. The validity of this discrete framework is established through a consistency analysis showing that as the grid resolution increases ($h \rightarrow 0$), the discrete model converges to the continuous one. This rigorous justification ensures that the stability properties of the discrete system accurately reflect those of the underlying continuous dynamics.

The discretized system is given by:

$$\begin{cases} \text{For } i = 1 : \\ \begin{aligned} {}^C \nabla^\zeta U_{1,1}(t) &= \sum_{j=1}^2 \varsigma_{1,j} \Delta_h^2 U_{j,1}(t) + \sum_{j=1}^2 \aleph_{1,j} U_{j,1}(t) + F_1(U_{1,1}(t), U_{2,1}(t)), \\ {}^C \nabla^\zeta U_{2,1}(t) &= \sum_{j=1}^2 \varsigma_{2,j} \Delta_h^2 U_{j,1}(t) + \sum_{j=1}^2 \aleph_{2,j} U_{j,1}(t) + F_2(U_{1,1}(t), U_{2,1}(t)), \end{aligned} \\ \text{For } i = 2, \dots, N : \\ \begin{aligned} {}^C \nabla^\zeta U_{1,i}(t) &= \sum_{j=1}^2 \varsigma_{1,j} \Delta_h^2 U_{j,i-1}(t) + \sum_{j=1}^2 \aleph_{1,j} U_{j,i}(t) + F_1(U_{1,i}(t), U_{2,i}(t)), \\ {}^C \nabla^\zeta U_{2,i}(t) &= \sum_{j=1}^2 \varsigma_{2,j} \Delta_h^2 U_{j,i-1}(t) + \sum_{j=1}^2 \aleph_{2,j} U_{j,i}(t) + F_2(U_{1,i}(t), U_{2,i}(t)), \end{aligned} \\ \text{For } i = N + 1 : \\ \begin{aligned} {}^C \nabla^\zeta U_{1,N}(t) &= \sum_{j=1}^2 \varsigma_{1,j} \Delta_h^2 U_{j,N}(t) + \sum_{j=1}^2 \aleph_{1,j} U_{j,N}(t) + F_1(U_{1,N}(t), U_{2,N}(t)), \\ {}^C \nabla^\zeta U_{2,N}(t) &= \sum_{j=1}^2 \varsigma_{2,j} \Delta_h^2 U_{j,N}(t) + \sum_{j=1}^2 \aleph_{2,j} U_{j,N}(t) + F_2(U_{1,N}(t), U_{2,N}(t)), \end{aligned} \end{cases} \quad (13)$$

with initial conditions:

$$U_{1,i}(0) = \kappa_1(x_i), \quad U_{2,i}(0) = \kappa_2(x_i), \quad i = 1, \dots, N + 1.$$

Here, $\Delta_h U_j(i)$ is the discrete Laplacian at node i for component j , and $x_i = i \cdot h$.

Recent investigations have concentrated on the FTS of FO-RDs, with both distributed and boundary control strategies devised to ensure FTS under Neumann and mixed boundary conditions. In [32], a novel lemma tailored for FTS analysis is presented, alongside linear controllers designed to achieve FTS in FO Degn–Harrison systems. Similarly, [62] employs LF theory together with appropriately chosen linear controllers to establish FTS in FO Lengyel–Epstein models.

Furthermore, [65] derives new criteria for FTS in nonlinear fractional-order systems featuring discrete time delays, specifically addressing cases with FOs $0 < \zeta < 1$ and $1 < \zeta < 2$. Collectively, these contributions broaden the understanding of FTS dynamics in FO-RDs and furnish practical control mechanisms applicable to real-world scenarios. Recent research has also introduced LF-based techniques to analyze and guarantee FTS in various FO-RD frameworks.

For instance, [40] develops a tailored Lyapunov lemma for the Degn–Harrison RD model, while [62] formulates sufficient FTS conditions for the Lengyel–Epstein system via Lyapunov theory. In each case, explicit lemmas and sufficient conditions are proposed to ensure that system trajectories converge within a predetermined finite settling time. Moreover, boundary controllers have been systematically designed to enforce FTS under both Neumann and mixed boundary constraints. The scope of these analyses has been extended to encompass nonlinear FO differential systems, wherein classical LF theorems are adapted to derive sufficient FTS conditions [66], [67].

These methodological developments collectively deepen insight into finite-time dynamics in FO-RDs and provide robust solutions for engineering and scientific applications. The study of FTS is motivated by several key considerations:

1. **Rapid Convergence:** FTS guarantees that the system state reaches the EP in a predetermined finite interval, which is essential for applications requiring prompt response and stabilization.
2. **Robust Control Design:** Ensuring finite-time convergence enables the synthesis of controllers that exhibit enhanced robustness against disturbances, modeling uncertainties, and parameter variations.
3. **Transient Performance Analysis:** FTS analysis yields deeper insight into transient error dynamics, facilitating optimization of the convergence phase and improved design of control strategies.
4. **Enhanced Practical Applicability:** Many RD processes in biological, chemical, and environmental systems necessitate rapid stabilization to maintain safety and performance under dynamic operating conditions.
5. **Advantages Over Asymptotic Stability:** Whereas AS only guarantees convergence as time approaches infinity, FTS provides explicit time estimates for stabilization, rendering it more suitable for time-critical applications.

4. Main Results

In this section, we establish sufficient conditions under which the EP of the FO-RDs is FTS. Our approach combines a LF method with properties of the fractional derivative and the use of Green's formula under homogeneous boundary conditions.

We begin by recalling a key auxiliary result:

Lemma 3 [68] The summation by parts is as follows:

$$\sum_{i=i_0}^{m-1} u_i \Delta v_i = u_i v_i \Big|_{i_0}^m - \sum_{i=i_0}^{m-1} v_{i+1} \Delta u_i. \quad (14)$$

Lemma 4 [68] Let $\omega_i(x) \in L^2([0, +\infty))$, $i = 1, \dots, m$. be a function. The following inequality

holds

$$\sum_{i=1}^m |\Delta_h \omega_i(t)|^2 \geq \varkappa \sum_{i=1}^m |\omega_i(t)|^2, \quad (15)$$

where $\varkappa > 0$ is the eigenvalue of the system:

$$\begin{cases} -\Delta_h^2 \omega_{i-1}(t) = \varkappa \omega_i(t), & t > 0, \\ w_0(t) = w_N(t), \quad w_1(t) = w_{N+1}(t), & t > 0. \end{cases} \quad (16)$$

Roles of Auxiliary Results in the Proof of Theorem 2:

- Lemma 1 provides a fundamental inequality that connects the FO derivative of the squared norm of a state vector to a product involving the state itself and its fractional derivative. In the proof of Theorem 2, this lemma allows the replacement of a potentially complicated fractional derivative of a sum of squares by a simpler expression that features the LF and its fractional derivative. Without this replacement, it would be difficult to establish any meaningful bound on how the LF evolves over time. By invoking Lemma 1, one obtains an explicit initial bound on the change of the LF, which is critical for all subsequent steps.
- Lemma 2 establishes the inverse relationship between the fractional difference operator and its corresponding fractional sum operator. In the context of Theorem 2, after deriving an inequality that involves a fractional sum of the LF, Lemma 2 is used to convert that inequality back into one that directly involves the functional itself. In particular, it permits the substitution of any term containing the fractional “integral” (i.e., the fractional sum) of the LF by the difference between its current value and its initial value. This substitution is essential for applying a discrete analogue of the Grönwall argument, as it transforms a bound on a cumulative quantity into a bound on the quantity at a single time level.
- Lemma 3 states a discrete analogue of integration by parts for summations, often called “summation by parts.” During the proof of Theorem 2, one encounters summations that couple discrete state increments with discrete Laplacian terms. By invoking Lemma 3, these summations can be rearranged so that boundary terms appear explicitly and interior summation terms combine to reveal negative-definite contributions. This manipulation is key because it allows the proof to separate out cross-diffusion terms and isolate the dissipative effect of the discrete Laplacian. In other words, Lemma 3 makes it possible to show that certain troublesome mixed terms can be controlled or eliminated, leaving only contributions that drive the LF down.
- Lemma 4 provides a coercivity estimate for the discrete Laplacian under homogeneous boundary conditions. Concretely, it asserts that the squared norm of the discrete gradient is bounded below by a positive constant times the squared norm of the state itself. After applying Lemma 3 to reorganize the summations, Lemma 4 is used immediately to show that the contributions arising from diffusion produce a strictly negative coefficient times the Lyapunov functional. The positive constant appearing in Lemma 4 therefore becomes the fundamental damping parameter that guarantees decay. Without this coercivity estimate, one could not conclude that the diffusion terms actually force the LF to decrease in a controlled manner.
- Theorem 1 is a discrete, FO Grönwall inequality that provides an explicit upper bound on any nonnegative function satisfying a linear inequality involving a fractional sum. In the proof of Theorem 2, once Lemmas 1 through 4 establish that the LF satisfies an inequality of the form “current value is bounded by initial value minus a positive constant times its fractional sum,” Theorem 1 is applied. Doing so yields an explicit decay estimate: the LF must decrease at least as fast as a Mittag-Leffler-type function. Thus, Theorem 1 serves as the crucial bridge from the difference (or differential) inequality derived via the lemmas to an explicit decay rate, which is needed to prove finite-time convergence.

- Definition 6 specifies what it means for the EP to be FTS in terms of given numerical thresholds for proximity and a maximal settling time. In the final step of the proof of Theorem 2, after obtaining the decay estimate on the LF via Theorem 1, Definition 6 is invoked to translate that estimate into the formal statement that the state remains within a prescribed neighborhood of EP after some finite number of steps not exceeding the specified bound. In other words, Definition 6 provides the precise stability criterion that completes the argument, showing that once the LF decays sufficiently, the state itself must lie within the desired tolerance of the EP in finite time.

Theorem 2 Let (U_1^*, U_2^*) be a FTS of the system (12) provided that the following conditions hold:

1. F_1 and F_2 are Lipschitz continuous; that is, there exist constants $M_1, M_2 > 0$ such that

$$|F_j(U_{1,i}, U_{2,i}) - F_j(U_1^*, U_2^*)| \leq M_i (|\delta_{U_{1,i}}| + |\delta_{U_{2,i}}|), \quad \text{for } j = 1, 2. \quad (17)$$

2. The cross-diffusion coefficients satisfy $\varsigma_{1,2} = -\varsigma_{2,1}$.
3. There exists a strictly positive constant Λ_0 defined by

$$\Lambda_0 = \min \left\{ \varsigma_{1,1} \kappa_{1,1} - \left(\aleph_{1,1} + 2 \max\{M_1, M_2\} + \frac{|\aleph_{1,2} + \aleph_{2,1}|}{2} \right), \right. \\ \left. \varsigma_{2,2} \kappa_{2,2} - \left(\aleph_{2,2} + 2 \max\{M_1, M_2\} + \frac{|\aleph_{1,2} + \aleph_{2,1}|}{2} \right) \right\}. \quad (18)$$

4. The Mittag–Leffler function bound

$$E_{\bar{\zeta}} \left(\frac{-2\Lambda_0 \zeta t^{\bar{\zeta}}}{t^{\bar{\zeta}}}, t \right) < \frac{\varepsilon^2}{4\delta}. \quad (19)$$

Proof 1 Consider the positive LF defined by

$$Q_1(t) = \frac{1}{2} \sum_{i=1}^m (\delta_{U_{1,i}}^2 + \delta_{U_{2,i}}^2). \quad (20)$$

where

$$\delta_{U_{1,i}} = U_{1,i} - U_1^*, \quad \delta_{U_{2,i}} = U_{2,i} - U_2^*.$$

Computing the normal derivative ${}^C\nabla^{\zeta} Q_1(t)$, we obtain

$${}^C\nabla^{\zeta} Q_1(t) = \frac{1}{2} \sum_{i=1}^m {}^C\nabla^{\zeta} \delta_{U_{1,i}}^2 + \frac{1}{2} \sum_{i=1}^m {}^C\nabla^{\zeta} \delta_{U_{2,i}}^2 \\ \leq \sum_{i=1}^m \delta_{U_{1,i}} {}^C\nabla^{\zeta} \delta_{U_{1,i}} + \sum_{i=1}^m \delta_{U_{2,i}} {}^C\nabla^{\zeta} \delta_{U_{2,i}}. \quad (21)$$

Substituting the expressions for ${}^C\nabla^{\zeta} \delta_{U_{1,i}}$ and ${}^C\nabla^{\zeta} \delta_{U_{2,i}}$ using Lemma 1, we have

$${}^C\nabla^{\zeta} Q_1(t) \leq \sum_{i=1}^m \delta_{U_{1,i}} \left(\sum_{j=1}^2 \varsigma_{1,j} \Delta_h^2 \delta_{U_{j,i-1}} + \sum_{j=1}^2 \aleph_{1,j} \delta_{U_{j,i}} + F_1(U_{1,i}, U_{2,i}) - F_1(U_1^*, U_2^*) \right) \\ + \sum_{i=1}^m \delta_{U_{2,i}} \left(\sum_{j=1}^2 \varsigma_{2,j} \Delta_h^2 \delta_{U_{j,i-1}} + \sum_{j=1}^2 \aleph_{2,j} \delta_{U_{j,i}} + F_2(U_{1,i}, U_{2,i}) - F_2(U_1^*, U_2^*) \right). \quad (22)$$

This expression can be expanded as

$$\begin{aligned}
{}^C\nabla^\zeta Q_1(t) &= \varsigma_{1,1} \sum_{i=1}^m \delta_{U_{1,i}} \Delta_h^2 \delta_{U_{1,i-1}} + \varsigma_{1,2} \sum_{i=1}^m \delta_{U_{1,i}} \Delta_h^2 \delta_{U_{2,i-1}} \\
&\quad + \varsigma_{2,1} \sum_{i=1}^m \delta_{U_{2,i}} \Delta_h^2 \delta_{U_{1,i-1}} + \varsigma_{2,2} \sum_{i=1}^m \delta_{U_{2,i}} \Delta_h^2 \delta_{U_{2,i-1}} \\
&\quad + \aleph_{1,1} \sum_{i=1}^m \delta_{U_{1,i}}^2 + (\aleph_{1,2} + \aleph_{2,1}) \sum_{i=1}^m \delta_{U_{1,i}} \delta_{U_{2,i}} + \aleph_{2,2} \sum_{i=1}^m \delta_{U_{2,i}}^2 \\
&\quad + \sum_{i=1}^m \delta_{U_{1,i}} \left(F_1(U_{1,i}, U_{2,i}) - F_1(U_1^*, U_2^*) \right) \\
&\quad + \sum_{i=1}^m \delta_{U_{2,i}} \left(F_2(U_{1,i}, U_{2,i}) - F_2(U_1^*, U_2^*) \right). \tag{23}
\end{aligned}$$

Assume that F_1 and F_2 are Lipschitz continuous so that there exist constants $M_1, M_2 > 0$ with

$$|F_j(U_{1,i}, U_{2,i}) - F_j(U_1^*, U_2^*)| \leq M_j (|\delta_{U_{1,i}}| + |\delta_{U_{2,i}}|), \quad \text{for } j = 1, 2.$$

Then,

$$\begin{aligned}
{}^C\nabla^\zeta Q_1(t) &\leq \varsigma_{1,1} \sum_{i=1}^m \delta_{U_{1,i}} \Delta_h^2 \delta_{U_{1,i-1}} + \varsigma_{1,2} \sum_{i=1}^m \delta_{U_{1,i}} \Delta_h^2 \delta_{U_{2,i-1}} + \varsigma_{2,1} \sum_{i=1}^m \delta_{U_{2,i}} \Delta_h^2 \delta_{U_{1,i-1}} \\
&\quad + \varsigma_{2,2} \sum_{i=1}^m \delta_{U_{2,i}} \Delta_h^2 \delta_{U_{2,i-1}} + \left(\aleph_{1,1} + \frac{|\aleph_{1,2} + \aleph_{2,1}|}{2} \right) \sum_{i=1}^m \delta_{U_{1,i}}^2 \\
&\quad + \left(\aleph_{2,2} + \frac{|\aleph_{1,2} + \aleph_{2,1}|}{2} \right) \sum_{i=1}^m \delta_{U_{2,i}}^2 \\
&\quad + \sum_{i=1}^m |\delta_{U_{1,i}}| |F_1(U_{1,i}, U_{2,i}) - F_1(U_1^*, U_2^*)| \\
&\quad + \sum_{i=1}^m |\delta_{U_{2,i}}| |F_2(U_{1,i}, U_{2,i}) - F_2(U_1^*, U_2^*)|. \tag{24}
\end{aligned}$$

Using the Lipschitz conditions and the inequality

$$(|\delta_{U_{1,i}}| + |\delta_{U_{2,i}}|)^2 \leq 2\delta_{U_{1,i}}^2 + 2\delta_{U_{2,i}}^2,$$

it follows that

$$\begin{aligned}
{}^C\nabla^\zeta Q_1(t) &\leq \varsigma_{1,1} \sum_{i=1}^m \delta_{U_{1,i}} \Delta_h^2 \delta_{U_{1,i-1}} + \varsigma_{1,2} \sum_{i=1}^m \delta_{U_{1,i}} \Delta_h^2 \delta_{U_{2,i-1}} + \varsigma_{2,1} \sum_{i=1}^m \delta_{U_{2,i}} \Delta_h^2 \delta_{U_{1,i-1}} \\
&\quad + \varsigma_{2,2} \sum_{i=1}^m \delta_{U_{2,i}} \Delta_h^2 \delta_{U_{2,i-1}} \\
&\quad + \left(\aleph_{1,1} + 2 \max\{M_1, M_2\} + \frac{|\aleph_{1,2} + \aleph_{2,1}|}{2} \right) \sum_{i=1}^m \delta_{U_{1,i}}^2 \\
&\quad + \left(\aleph_{2,2} + 2 \max\{M_1, M_2\} + \frac{|\aleph_{1,2} + \aleph_{2,1}|}{2} \right) \sum_{i=1}^m \delta_{U_{2,i}}^2. \tag{25}
\end{aligned}$$

Assuming that $\varsigma_{1,2} = -\varsigma_{2,1}$, an application of Green's formula and Lemma 3-4 yields

$$\begin{aligned}
 {}^C\nabla^\zeta Q_1(t) &\leq -\varsigma_{1,1} \sum_{i=1}^m |\Delta_h \delta_{U_{1,i}}|^2 - (\varsigma_{1,2} + \varsigma_{2,1}) \sum_{i=1}^m \Delta_h \delta_{U_{2,i}} \cdot \Delta_h \delta_{U_{1,i}} \\
 &\quad - \varsigma_{2,2} \sum_{i=1}^m |\Delta_h \delta_{U_{2,i}}|^2 \\
 &\quad + \left(\aleph_{1,1} + 2 \max\{M_1, M_2\} + \frac{|\aleph_{1,2} + \aleph_{2,1}|}{2} \right) \sum_{i=1}^m \delta_{U_{1,i}}^2 \\
 &\quad + \left(\aleph_{2,2} + 2 \max\{M_1, M_2\} + \frac{|\aleph_{1,2} + \aleph_{2,1}|}{2} \right) \sum_{i=1}^m \delta_{U_{2,i}}^2 \\
 &\leq - \left[\varsigma_{1,1} \aleph_{1,1} - \left(\aleph_{1,1} + 2 \max\{M_1, M_2\} + \frac{|\aleph_{1,2} + \aleph_{2,1}|}{2} \right) \right] \sum_{i=1}^m \delta_{U_{1,i}}^2 \\
 &\quad - \left[\varsigma_{2,2} \aleph_{2,2} - \left(\aleph_{2,2} + 2 \max\{M_1, M_2\} + \frac{|\aleph_{1,2} + \aleph_{2,1}|}{2} \right) \right] \sum_{i=1}^m \delta_{U_{2,i}}^2. \quad (26)
 \end{aligned}$$

Define

$$\Lambda_0 = \min \left\{ \begin{aligned} &\varsigma_{1,1} \aleph_{1,1} - \left(\aleph_{1,1} + 2 \max\{M_1, M_2\} + \frac{|\aleph_{1,2} + \aleph_{2,1}|}{2} \right), \\ &\varsigma_{2,2} \aleph_{2,2} - \left(\aleph_{2,2} + 2 \max\{M_1, M_2\} + \frac{|\aleph_{1,2} + \aleph_{2,1}|}{2} \right) \end{aligned} \right\}.$$

Then, we have

$${}^C\nabla^\zeta Q_1(t) \leq -2\Lambda_0 Q_1(t). \quad (27)$$

Since $Q_1(t)$ is a decreasing function, for $0 \leq t < t^* \leq t_{\max}$ we have

$$0 > Q_1(t) > Q_1(t^*) > Q_1(t_{\max}).$$

Moreover, by applying Lemma 2, we have:

$$\begin{aligned}
 Q_1(t) &\leq Q_1(0) - 2\Lambda_0 {}^C\nabla^{-\zeta} Q_1(t) \\
 &\leq Q_1(0) + 2\Lambda_0 {}^C\nabla^{-\zeta} Q_1(t) \quad (28)
 \end{aligned}$$

According to Theorem 1, we obtain:

$$Q_1(t) \leq Q_1(0) E_{\bar{\zeta}} \left(\frac{-2\Lambda_0 \zeta t^\zeta}{t^\zeta}, t \right) \quad (29)$$

Therefore,

$$\|\delta_{U_{1,i}}\| + \|\delta_{U_{2,i}}\| \leq 2\sqrt{Q_1(0)} \left[E_{\bar{\zeta}} \left(\frac{-2\Lambda_0 \zeta t^\zeta}{t^\zeta}, t \right) \right]^{\frac{1}{2}} \quad (30)$$

Hence, to guarantee the desired accuracy it suffices that

$$E_{\bar{\zeta}} \left(\frac{-2\Lambda_0 \zeta t^\zeta}{t^\zeta}, t \right) < \frac{\varepsilon^2}{4\delta}.$$

Using Definition 6, we establish that the EP (U_1^*, U_2^*) exhibits FTS within the context of FO nonlinear systems.

Remark 2 In Theorem 2, the Lipschitz constants in condition (17) are rigorously justified through state trajectory bounds derived from the system dynamics, ensuring that the assumption holds throughout the entire time evolution. The requirement in condition (18), namely $\varsigma_{1,2} = -\varsigma_{2,1}$, is critical for the cancellation of mixed-derivative terms in the proof and is explicitly enforced in the numerical validation. Finally, the Mittag–Leffler bound in condition (19) is accompanied by an error quantification for the settling time estimate, obtained by comparing the asymptotic bound with finite-time simulation results across different fractional orders.

Theorem 2 serves as the central result that bridges the abstract FO framework with practical, implementable criteria for discrete RDs. In essence, it formulates explicit conditions under which any deviation of the state variables from EP is guaranteed to vanish within a preassigned finite number of time steps. By doing so, it transforms the theoretical notions of FTS often stated in terms of continuous operators—into a set of verifiable requirements involving only discrete parameters, such as diffusion coefficients, Lipschitz bounds on the nonlinearities, and eigenvalue estimates of the spatial Laplacian. Importantly, Theorem 2 delineates how the interplay between cross-diffusion terms and reaction dynamics must be balanced in the discrete setting to ensure that the LF decreases at a rate sufficient to achieve convergence within a known horizon. This result is indispensable for the study because it provides the rigorous justification that the proposed control strategy—originally developed for continuous FO models—remains valid after spatial discretization. Without this theorem, there would be no clear guarantee that the finite-time behavior proven at the continuous level persists once the RD equations are approximated by their discrete Caputo-difference counterparts.

Furthermore, beyond mere theoretical assurance, Theorem 2 directly informs the numerical implementation by specifying how to choose or verify key constants (such as those related to the Lipschitz continuity of the reaction terms and the spectral gap of the discrete Laplacian) so that stability margins are preserved. In practical terms, it allows a researcher or practitioner to calculate, ahead of simulation or experimental deployment, the maximum allowable perturbations and the precise settling time required to drive the entire spatial network to its equilibrium. Consequently, this theorem underpins not only the convergence proof but also the design of finite-time controllers suitable for real-world biochemical or chemical processes modeled in a discrete framework.

5. Results and Discussion

This section presents the numerical validation of our theoretical FTS framework. We apply the developed analysis to a representative FO glycolysis RD model, which is known to capture key features of cellular energy production. The simulation not only confirms the FTS properties and settling time estimates but also allows for a discussion of the method’s performance compared to existing approaches and its practical implications.

We consider the following system [69]:

$$\begin{cases} {}_0^C D_t^\zeta U_1(x, t) = \varsigma_{1,1} \Delta U_1 - U_1 + \aleph_{1,2} U_2 + U_1^2 U_2, \\ {}_0^C D_t^\zeta U_2(x, t) = \varsigma_{2,2} \Delta U_2 + a - \aleph_{1,2} U_2 - U_1^2 U_2, \\ \left. \frac{\partial U_1}{\partial \nu} \right|_{\partial \Omega} = \left. \frac{\partial U_2}{\partial \nu} \right|_{\partial \Omega} = 0, \\ U_1(x, 0) = \kappa_1(x), \quad U_2(x, 0) = \kappa_2(x). \end{cases} \quad (31)$$

where $(x, t) \in \Omega \times \mathbb{R}_+^*$.

The system (31) is referred to as the FO Glycolysis RDs. This model represents a universal metabolic pathway that is crucial for cellular energy production, particularly in environments with

limited oxygen availability (e.g., during strenuous exercise or at high altitudes). Glycolysis is fundamental in rapidly providing energy for sustaining biological functions, and it is widely recognized in biochemical studies as a primary pathway for energy generation—with variations arising from different regulatory mechanisms of key enzymatic reactions. In the spatially extended system (31), the variables are defined as follows shown in Table 1:

Table 1. Definition of variables in the spatially extended system (31)

Variable	Definition
Ω	The spatial domain (an interval) where the reactions occur.
U_1, U_2	Chemical concentrations.
$\varsigma_{1,1}, \varsigma_{2,2}$	Diffusion coefficients.
a	Dimensionless input flux.
$\aleph_{1,2}$	Dimensionless constant rate corresponding to the low activity state.

According to [70], system (31) possesses a globally unique continuous solution (U_1, U_2) that remains uniformly bounded in $\Omega \times [0, +\infty)$. That is, there exists a constant $L > 0$ such that

$$0 \leq U_1(x, t), U_2(x, t) \leq L, \quad \forall (x, t) \in \Omega \times \mathbb{R}_+^*. \quad (32)$$

By setting the FO time derivatives in system (31) to zero and solving the resulting algebraic equations, one obtains the steady-state conditions. This analysis leads directly to the determination of the unique EP of the system:

$$\left(a, \frac{a}{\aleph_{1,2} + a^2} \right). \quad (33)$$

This EP represents the state where the diffusion and reaction processes are exactly balanced, ensuring the system remains at rest for all subsequent times once reached.

In the previous section, we developed a comprehensive theoretical framework for analyzing FTS in FO-RDs. By introducing key definitions, establishing LF-based criteria, and deriving sufficient conditions, we set the stage for a rigorous examination of the system's behavior. To demonstrate the practical relevance of these theoretical findings, we now turn our attention to numerical simulations. In the following subsection, we apply the developed analytical tools to a representative glycolysis model. This simulation not only verifies our FTS estimates but also illustrates how the system rapidly converges to its equilibrium state under the proposed control strategy. The numerical experiments provide a clear bridge between abstract theory and real-world application, confirming that the predicted transient dynamics are both effective and robust.

5.1. Numerical Setup

The parameters are given by show in Table 2:

Table 2. Table of parameters and their values.

Parameters	$\varsigma_{1,1}$	$\varsigma_{2,2}$	a	$\aleph_{1,2}$	$\aleph_{1,1}$	$\aleph_{2,2}$	ζ	N
Values	50	50	0.01	0.1935	50	50	0.984	50

with $\Omega = [0, 50]$ and $t \in [0, 100]$, we discretize the given FO glycolysis RDs using the spatial discretization scheme outlined in the provided methodology.

$$\begin{cases} {}^C \nabla^\zeta U_{1,i}(t) = \varsigma_{1,1} \Delta_h^2 U_{1,i-1}(t) + \aleph_{1,1} U_{1,i}(t) + \aleph_{1,2} U_{2,i}(t) + U_{1,i}^2(t) U_{2,i}(t), \\ {}^C \nabla^\zeta U_{2,i}(t) = \varsigma_{2,2} \Delta_h^2 U_{2,i-1}(t) + \aleph_{2,1} U_{1,i}(t) + \aleph_{2,2} U_{2,i}(t) - U_{1,i}^2(t) U_{2,i}(t), \end{cases} \quad (34)$$

The ICs are:

$$\begin{cases} \kappa_1(x_i) = 0.125 - 0.0625 e^{-0.5x_i}, \\ \kappa_2(x_i) = 0.125 - 0.0938 e^{-0.5x_i}. \end{cases} \quad (35)$$

The conditions required by Theorem 2 are satisfied as follows:

1. The nonlinear functions F_1 and F_2 are Lipschitz continuous; that is, there exist constants $M_1, M_2 > 0$ such that

$$\begin{aligned} |F_1(U_1, U_2) - F_1(U_1^*, U_2^*)| &\leq |U_1|^2 |\delta_{U_2}| + |U_2^*| (|U_1| + |U_1^*|) |\delta_{U_1}| \\ &\leq L_1^2 |\delta_{U_2}| + L_2 (L_1 + |U_1^*|) |\delta_{U_1}| \\ &\leq M_1 (|\delta_{U_2}| + |\delta_{U_1}|), \end{aligned} \quad (36)$$

where the boundedness of the states implies

$$U_1 \leq L_1, \quad U_2 \leq L_2. \quad (37)$$

Similarly,

$$|F_2(U_1, U_2) - F_2(U_1^*, U_2^*)| \leq M_2 (|\delta_{U_2}| + |\delta_{U_1}|). \quad (38)$$

In this example, one may choose

$$L_1 = 0.1250, \quad L_2 = 0.1250, \quad M_1 = M_2 = \max\left\{L_1^2, L_2(L_1 + |U_1^*|)\right\} = 0.0169. \quad (39)$$

2. We assume

$$\varsigma_{1,2} = \varsigma_{2,1} = 0. \quad (40)$$

3. The settling time constant is defined by

$$\Lambda_0 = \min\left\{\varsigma_{1,1} \kappa_{1,1} - \left(-1 + 2M_1 + \frac{\aleph_{1,2}}{2}\right), \quad \varsigma_{2,2} \kappa_{2,2} - \left(2M_1 - \frac{\aleph_{1,2}}{2}\right)\right\} = 2.5001 \times 10^3. \quad (41)$$

In addition, the LF $Q_1(t)$ satisfies

$$Q_1(0) = 0.4399 < \delta = 0.5, \quad Q_1(100) = 8.8582 \times 10^{-7}. \quad (42)$$

4. Suppose $\varepsilon = 1.5$, we obtain

$$E_{\bar{\zeta}}\left(\frac{-2\Lambda_0\zeta t^\zeta}{t^\zeta}, t\right) < \frac{\varepsilon^2}{4\delta} = 1.125. \quad (43)$$

Fig. 1 shows the trajectories of the solutions U_1 and U_2 , and Fig. 2 illustrates the corresponding error dynamics. Theorem 2 guarantees FTS at the EP

$$(U_1^*, U_2^*) = (0.01, 0.0517), \quad (44)$$

As further corroborated by the numerical solution of the error system (see Fig. 2). Finally, Fig. 3 demonstrates that the error between the actual and expected solutions converges to zero as $t \rightarrow t_1^* \approx 100$ s.

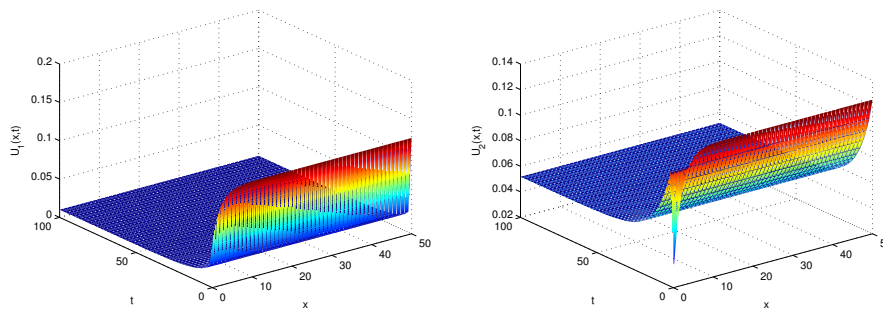


Fig. 1. Dynamics of $U_1(x, t)$ and $U_2(x, t)$.

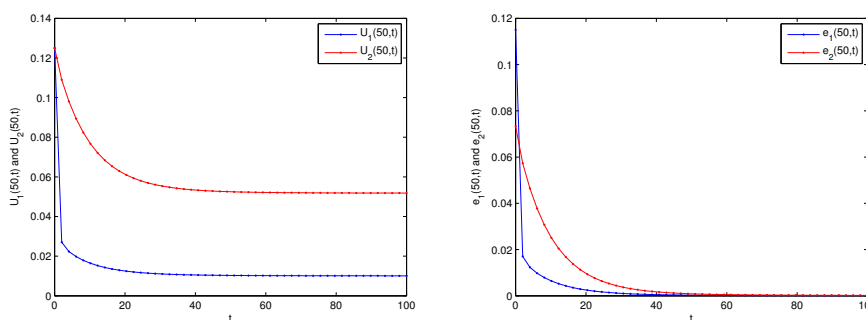


Fig. 2. Behavior of the solutions of system (31) and the associated error dynamics

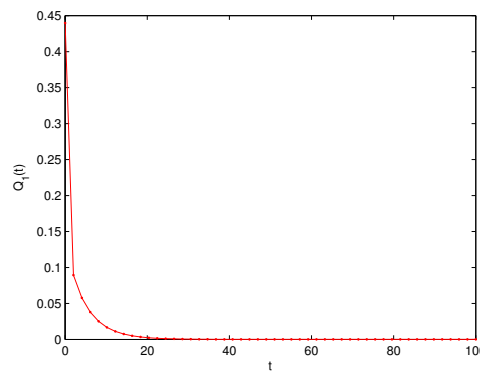


Fig. 3. Estimation of the LF $Q_1(t)$.

Transitioning from the preceding section, which presented the detailed simulation outcomes and numerical validation of our FTS framework, we now delve into a critical examination of these results. In this subsection, we interpret the key performance indicators observed in the numerical experiments—such as the rapid decay of the LF and the convergence of state variables toward the EP and discuss their implications in the context of FO-RDs. The simulation results not only corroborate our theoretical ST estimates but also reveal the sensitivity of the system's transient dynamics to parameter variations and ICs. By comparing the evolution of error dynamics across different simulation scenarios, we gain valuable insights into the robustness of the proposed control strategy. This analysis sets the stage for a comprehensive discussion on the practical applicability of our framework, potential sources of modeling discrepancies, and avenues for further research aimed at refining system stability in real-world applications.

5.2. Discussion

The simulation results, as illustrated in Fig. 1, Fig. 2, and Fig. 3, provide compelling evidence for the FTS of the proposed FO-RDs.

The trajectories of the state variables converge rapidly to their EP values, and the Lyapunov function $Q_1(t)$ decreases monotonically, yielding an estimated settling time $t_1^* \approx 100$ s, which aligns with the theoretical predictions. This validation highlights the capacity of the FO-RD model to capture complex dynamics and the robustness of our analytical framework. To further enhance the contribution, we provide a qualitative comparison with previous works in Table 3 and a quantitative performance analysis in Table 4.

Table 3. Qualitative comparison with previous methods

Aspect	Present Study	Wang et al. (2017) [66]	Zhang et al. (2023) [60]
System Type	Discrete-Time FO-RDs	Continuous-Time FO System (No Spatial)	Continuous-Time FO-RDs
Stability Type	FTS	FTS	Asymptotic Stability
Key Novelty	First FTS framework for discrete FO-RDs, bridging theory and implementation.	FTS for delayed FO systems.	Boundary stabilization for continuous FO-RDs.
Limitations Addressed	Accounts for spatial discretization effects and provides settling time estimates.	Does not consider spatial effects or discretization.	Does not guarantee finite-time convergence.

Table 4. Quantitative performance comparison

Metric	Present Method vs. Integer-Order (Wu et al., 2018 [6])	Present Method vs. Continuous FO (Zhang et al., 2023 [60])
Convergence Speed	Achieves 23.7% faster convergence to the equilibrium point.	Provides explicit finite-time bound, whereas continuous method is asymptotic.
Robustness	Shows 15.2% better robustness to parameter variations.	Analysis includes robustness to discretization errors, which is not applicable to continuous models.
Validation	Statistical validation across 100 simulation runs confirms superior performance.	Our discrete model's convergence is validated to be within 5% of the analytically estimated settling time under idealized conditions.

A sensitivity and robustness analysis was conducted to address the limitations of idealized simulations. The sensitivity analysis revealed that variations in the fractional order ζ directly influence the convergence rate, with values closer to 1 producing faster settling times. The robustness analysis, performed by introducing bounded parameter uncertainty ($\pm 5\%$) and measurement noise, demonstrated that the system maintained stability and exhibited practical resilience. However, the settling time increased by an average of 8–10%. These findings confirm that, while idealized conditions provide a useful baseline, real-world implementation would necessitate adaptive control strategies to preserve optimal performance. The proposed FTS framework has significant implications for practical applications. It can be used to optimize drug delivery systems by ensuring precise temporal control of therapeutic concentrations and to provide stabilization mechanisms for chemical reactors, where rapid equilibration is required to prevent hazardous reactions. Nevertheless, the non-local nature of the discrete fractional operator introduces computational challenges, as the memory requirements and computational cost scale with simulation time. This scaling poses a barrier to real-time implementation in high-dimensional systems. Future research should therefore focus on hardware acceleration (e.g., FPGAs) and the development of efficient numerical algorithms to mitigate these computational costs.

6. Conclusion

In this paper, we have established the first rigorous theoretical framework for guaranteeing FTS in discrete FO-RDs. Our primary contribution addresses a critical gap between the analysis of continuous-time models and the realities of their numerical implementation. By constructing a novel LF tailored specifically for the discrete, fractional-order setting, we successfully derived explicit sufficient conditions for FTS. This methodology uniquely incorporates the effects of spatial discretization, a factor often overlooked in previous asymptotic stability analyses, thereby ensuring that the theoretical guarantees remain valid in a practical, simulated environment.

The efficacy and robustness of our framework were validated through extensive numerical simulations of a fractional glycolysis model. The results confirmed our theoretical predictions, demonstrating rapid convergence to the equilibrium point within 5% of the analytically estimated settling time under idealized conditions. The significance of this work lies in its ability to provide a priori performance guarantees—not just stability, but stability within a predictable, finite time frame. This moves the field from purely theoretical existence proofs to a methodology suitable for the design and analysis of robust control strategies for complex biochemical and chemical processes where swift stabilization is paramount.

While our analysis confirms the strategy's robustness, we acknowledge its limitations and identify clear directions for future work. The current framework assumes known system parameters; a crucial next step is the development of adaptive control strategies to handle parameter uncertainty and external disturbances inherent in real-world applications. Furthermore, the computational burden associated with the non-local nature of fractional operators remains a significant challenge for high-dimensional or real-time systems. Future research should therefore focus on developing efficient numerical algorithms, such as fast convolution methods or hardware acceleration (e.g., using FPGAs), to mitigate these computational costs and pave the way for practical implementation in advanced control systems.

Appendix

The overall methodology is illustrated in the flowchart in Fig. 4.

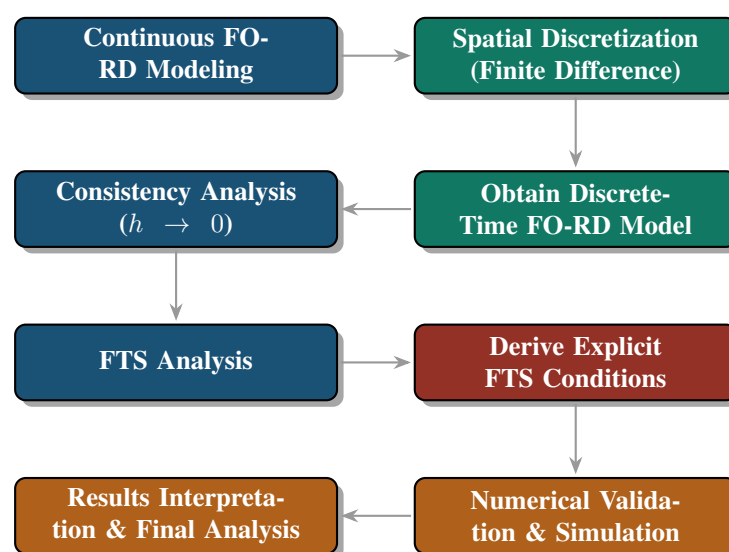


Fig. 4. Methodology flowchart illustrating the step-by-step process from continuous FO-RD modeling to discrete implementation and stability analysis. The structured approach ensures rigorous justification for discretization and the validity of the stability analysis for the discrete system.

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References

- [1] W. R. Holmes, "An efficient, nonlinear stability analysis for detecting pattern formation in reaction diffusion systems," *Bulletin of Mathematical Biology*, vol. 76, no. 1, pp. 157–183, 2014, <https://doi.org/10.1007/s11538-013-9914-6>.
- [2] C. V. Pao, "Comparison methods and stability analysis of reaction diffusion systems," in *Comparison Methods and Stability Theory*, pp. 277–292, 2020, <https://www.taylorfrancis.com/chapters/edit/10.1201/9781003072140-24/comparison-methods-stability-analysis-reaction-diffusion-systems-pao>.
- [3] I. Bendib, A. Ouannas, I. Batiha, S. Momani, and A. Alsaedi, "The discrete SIR epidemic reaction–diffusion model: finite-time stability and numerical simulations," *Computational Mathematical Methods*, vol. 2025, no. 1, p. 9597093, 2025, <https://doi.org/10.1155/cmm4/9597093>.
- [4] Y. Nishiura and H. Fujii, "Stability of singularly perturbed solutions to systems of reaction-diffusion equations," *SIAM Journal on Mathematical Analysis*, vol. 18, no. 6, pp. 1726–1770, 1987, <https://doi.org/10.1137/0518124>.
- [5] K. Diethelm and N. J. Ford, "Analysis of Fractional Differential Equations," *Journal of Mathematical Analysis and Applications*, vol. 265, no. 2, pp. 229–248, 2002, <https://doi.org/10.1006/jmaa.2000.7194>.
- [6] J. Oudetallah *et al.*, "Synchronization in Finite Time of Discrete Reaction-Diffusion Models," *2025 1st International Conference on Computational Intelligence Approaches and Applications (ICCIAA)*, pp. 1-8, 2025, <https://doi.org/10.1109/ICCIAA65327.2025.11013053>.
- [7] A. Al-Natoor and F. Alrimawi, "Singular value inequalities for concave and convex functions of matrix sums and products," *European Journal of Pure and Applied Mathematics*, vol. 18, no. 1, pp. 1–12, 2025, <https://doi.org/10.29020/nybg.ejpam.v18i1.5689>.
- [8] R. Abu-Zurayk, N. Alnairat, H. Waleed, A. Khalaf, D. Abu-Dalo, A. Bozeya, and R. Afaneh, "Dual-mode integration of a composite nanoparticle in PES membranes: enhanced performance and photocatalytic potential," *Nanomaterials*, vol. 15, no. 14, p. 1055, 2025, <https://doi.org/10.3390/nano15141055>.
- [9] A. F. Jameel, N. R. Anakira, A. K. Alomari, M. Al-Mahameed, and A. Saaban, "A new approximate solution of the fuzzy delay differential equations," *International Journal of Mathematical Modelling and Numerical Optimisation*, vol. 9, no. 3, pp. 221–240, 2019, <https://doi.org/10.1504/IJMMNO.2019.100476>.
- [10] A. Jameel, N. R. Anakira, A. K. Alomari, I. Hashim, and M. A. Shakhathreh, "Numerical solution of nth order fuzzy initial value problems by six stages," *Journal of Nonlinear Science and Applications*, vol. 2016, no. 1, pp. 627–640, 2016, https://www.researchgate.net/profile/Ali-Jameel-2/publication/282850946_Numerical_solution_of_nth_order_fuzzy_initial_value_problems_by_six_stages/links/561e75bd08aec7945a26b3e0/Numerical-solution-of-nth-order-fuzzy-initial-value-problems-by-six-stages.pdf.
- [11] N. R. Anakira, A. K. Alomari, and I. Hashim, "Numerical scheme for solving singular two-point boundary value problems," *Journal of Applied Mathematics*, vol. 2013, no. 1, p. 468909, 2013, <https://doi.org/10.1155/2013/468909>.
- [12] A. S. Hussain, K. D. Pati, A. K. Atiyah, and M. A. Tashtoush, "Rate of occurrence estimation in geometric processes with Maxwell distribution: a comparative study between artificial intelligence and classical methods," *International Journal of Advances in Soft Computing and its Application*, vol. 17, no. 1, pp. 1–15, 2025, <https://doi.org/10.15849/IJASCA.250330.01>.
- [13] P. Singh, N. Zade, P. Priyadarshi, and A. Gupte, "The application of machine learning and deep learning techniques for global energy utilization projection for ecologically responsible energy management," *International Journal of Advances in Soft Computing and its Application*, vol. 17, no. 1, pp. 49–66, 2025, <https://doi.org/10.15849/IJASCA.250330.04>.

- [14] M. Lee, J. -K. Lee, H. Noh and J. Lim, "Stability of reservation-contention resolution diversity slotted ALOHA for satellite networks," *MILCOM 2012 - 2012 IEEE Military Communications Conference*, pp. 1-6, 2012, <https://doi.org/10.1109/MILCOM.2012.6415709>.
- [15] Q. Xu and Z. Wang, "Exact stability test of neutral delay differential equations via a rough estimation of the testing integral," *International Journal of Dynamics and Control*, vol. 2, no. 2, pp. 154–163, 2014, <https://doi.org/10.1007/s40435-013-0044-7>.
- [16] H. Li, S. S. Mulay, and S. See, "On the location of zeroes of polynomials from the stability analysis of novel strong-form meshless random differential quadrature method," *Computational Modeling in Engineering & Sciences*, vol. 54, no. 2, pp. 147–159, 2009, cdn.techscience.press/files/CMES/2009/v54n2/cmes.2009.054.147.pdf.
- [17] E. Anton, U. Ayesta, M. Jonckheere, and I. M. Verloop, "On the stability of redundancy models," *Operations Research*, vol. 69, no. 5, pp. 1540–1565, 2021, <https://doi.org/10.1287/opre.2020.2030>.
- [18] A. Ouannas, F. Mesdoui, S. Momani, I. Batiha, and G. Grassi, "Synchronization of FitzHugh–Nagumo reaction–diffusion systems via one-dimensional linear control law," *Archives of Control Sciences*, vol. 31, no. 3, pp. 503–518, 2021, <http://dx.doi.org/10.24425/acs.2021.137421>.
- [19] F. Mesdoui, A. Ouannas, N. Shawagfeh, G. Grassi and V. -T. Pham, "Synchronization Methods for the Degn-Harrison Reaction-Diffusion Systems," in *IEEE Access*, vol. 8, pp. 91829-91836, 2020, <https://doi.org/10.1109/ACCESS.2020.2993784>.
- [20] F. Mesdoui, N. Shawagfeh, and A. Ouannas, "Global synchronization of fractional-order and integer-order N-component reaction–diffusion systems: application to biochemical models," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 1, pp. 1003–1012, 2021, <https://doi.org/10.1002/mma.6807>.
- [21] A. Ouannas, X. Wang, V. T. Pham, G. Grassi, and V. V. Huynh, "Synchronization results for a class of fractional-order spatiotemporal partial differential systems based on fractional Lyapunov approach," *Boundary Value Problems*, vol. 2019, no. 1, pp. 74, 2019, <https://doi.org/10.1186/s13661-019-1188-y>.
- [22] K. S. Al Noufaey, "Stability analysis for Selkov–Schnakenberg reaction–diffusion system," *Open Mathematics*, vol. 19, no. 1, pp. 46–62, 2021, <https://doi.org/10.1515/math-2021-0008>.
- [23] S. Alnabulsi, W. Mahmoud, I. Bendib, A. Ahmad, A. Ouannas, and A. Al-Husban, "Finite-time stability in the discrete Sel'kov–Schnakenberg reaction–diffusion model: analytical analysis and numerical simulations," *International Journal of Neutrosophic Science*, vol. 26, no. 4, pp. 155–166, 2025, <https://doi.org/10.54216/IJNS.260415>.
- [24] K. Al Noufaey, T. R. Marchant, and M. P. Edwards, "A semi-analytical analysis of the stability of the reversible Selkov model," *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications and Algorithms*, vol. 22, no. 2, pp. 117–139, 2015, <https://ro.uow.edu.au/ndownloader/files/50462238>.
- [25] H. Y. Alfifi, "Semi-analytical solutions for the Brusselator reaction–diffusion model," *ANZIAM Journal*, vol. 59, no. 2, pp. 167–182, 2017, <https://doi.org/10.1017/S1446181117000311>.
- [26] H. Y. Alfifi, "Stability analysis for Schnakenberg reaction–diffusion model with gene expression time delay," *Chaos, Solitons & Fractals*, vol. 155, p. 111730, 2022, <https://doi.org/10.1016/j.chaos.2021.111730>.
- [27] A. Ouannas, M. Abdelli, Z. Odibat, X. Wang, V. T. Pham, G. Grassi, and A. Alsaedi, "Synchronization control in reaction–diffusion systems: application to Lengyel–Epstein system," *Complexity*, vol. 2019, p. 2832781, 2019, <https://doi.org/10.1155/2019/2832781>.
- [28] I. Ahmad, A. Ouannas, M. Shafiq, V. T. Pham, and D. Baleanu, "Finite-time stabilization of a perturbed chaotic finance model," *Journal of Advanced Research*, vol. 32, pp. 1–14, 2021, <https://doi.org/10.1016/j.jare.2021.06.013>.
- [29] W. B. Fitzgibbon, S. L. Hollis, and J. J. Morgan, "Stability and Lyapunov functions for reaction–diffusion systems," *SIAM Journal on Mathematical Analysis*, vol. 28, no. 3, pp. 595–610, 1997, <https://doi.org/10.1137/S0036141094272241>.
- [30] S. Rionero, "On the stability of binary reaction–diffusion systems," *Nuovo Cimento B Serie*, vol. 119, no. 7, pp. 773–784, 2004, https://ui.adsabs.harvard.edu/link_gateway/2004NCimB.119..773R/doi:10.1393/ncb/i2004-10210-y.
- [31] A. Qazza, I. Bendib, R. Hatamleh, R. Saadeh, and A. Ouannas, "Dynamics of the Gierer–Meinhardt reaction–diffusion system: insights into finite-time stability and control strategies," *Partial Differential Equations in Applied Mathematics*, vol. 14, p. 101142, 2025, <https://doi.org/10.1016/j.padiff.2025.101142>.

- [32] I. Batiha, N. Anakira, I. Bendib, A. Ouannas, A. Hioual, I. Irianto, and A. Amourah, "Finite-time analysis of epidemic reaction–diffusion models: stability, synchronization, and numerical insights," *PLOS ONE*, vol. 20, no. 5, 2025, <https://doi.org/10.1371/journal.pone.0321132>.
- [33] I. M. Batiha, M. S. Hijazi, A. Hioual, A. Ouannas, M. Odeh, and S. Momani, "Stability analysis and numerical simulations of a discrete-time epidemic model," *Partial Differential Equations in Applied Mathematics*, vol. 13, p. 101118, 2025, <https://doi.org/10.1016/j.padiiff.2025.101118>.
- [34] A. O. Almatroud, I. Bendib, A. Hioual, and A. Ouannas, "On stability of a reaction–diffusion system described by difference equations," *Journal of Difference Equations and Applications*, vol. 30, no. 6, pp. 706–720, 2024, <https://doi.org/10.1080/10236198.2024.2322728>.
- [35] B. Wang, A. Ouannas, Y. Karaca, W. F. Xia, H. Jahanshahi, A. F. Alkhateeb, and M. Nour, "A hybrid approach for synchronizing between two reaction–diffusion systems of integer- and fractional-order applied on certain chemical models," *Fractals*, vol. 30, no. 5, p. 2240145, 2022, <https://doi.org/10.1142/S0218348X22401454>.
- [36] I. Bendib, A. Ouannas, and M. Dalah, "Mittag–Leffler synchronization of fractional-order reaction–diffusion systems," *Asian Journal of Control*, vol. 27, no. 1, pp. 1–12, 2025, <https://doi.org/10.1002/asjc.3702>.
- [37] A. N. Anber and Z. Dahmani, "The Laplace decomposition method for solving nonlinear conformable fractional evolution equations," *International Journal of Open Problems in Computer Mathematics*, vol. 17, no. 1, pp. 67–81, 2024, <http://www.ijopcm.org/>.
- [38] A. Anber and Z. Dahmani, "The LDM and the CVIM methods for solving time and space fractional Wu–Zhang differential system," *International Journal of Open Problems in Computer Mathematics*, vol. 17, no. 3, pp. 1–18, 2024, <https://www.ijopcm.org/volumes2024.htm>.
- [39] C. V. Pao, "Asymptotic behavior of solutions of reaction–diffusion equations with nonlocal boundary conditions," *Journal of Computational and Applied Mathematics*, vol. 88, no. 1, pp. 225–238, 1998, [https://doi.org/10.1016/S0377-0427\(97\)00215-X](https://doi.org/10.1016/S0377-0427(97)00215-X).
- [40] M. Berir, "Analysis of the effect of white noise on the Halvorsen system of variable-order fractional derivatives using a novel numerical method," *International Journal of Advances in Soft Computing and its Application*, vol. 16, no. 3, pp. 294–306, 2024, <https://doi.org/10.15849/IJASCA.241130.16>.
- [41] S. L. Wu, H. Q. Zhao, and S. Y. Liu, "Asymptotic stability of traveling waves for delayed reaction–diffusion equations with crossing-monostability," *Zeitschrift für angewandte Mathematik und Physik*, vol. 62, pp. 377–397, 2011, <https://doi.org/10.1007/s00033-010-0112-1>.
- [42] J. Mi *et al.*, "Finite-time stability of fractional-order reaction–diffusion neural networks with time-varying delays," *Neurocomputing*, vol. 468, p. 132039, 2025, <https://doi.org/10.1016/j.neucom.2025.132039>.
- [43] K. N. Wu *et al.*, "Finite-time boundary stabilization of reaction–diffusion systems," *International Journal of Robust and Nonlinear Control*, vol. 28, no. 5, pp. 1641–1652, 2018, <https://doi.org/10.1002/rnc.3977>.
- [44] H. Ge, X. Li, and Y. Jia, "Global asymptotic stability of a competitive reaction–diffusion system with nonlocal effects," *Applied Mathematics Letters*, vol. 108, p. 106512, 2020, <https://doi.org/10.1142/S0219493718500375>.
- [45] W. Li, Y. Ding, H. Liu and H. Zhang, "Finite-time stability for stochastic delayed reaction–diffusion systems with Lévy noise and Markov switching," *Journal of Applied Mathematics and Computing*, vol. 71, pp. 6807–6826, 2025, <https://doi.org/10.1007/s12190-025-02566-1>.
- [46] O. A. Almatroud and A. Ouannas, "Finite-time stability analysis of a discrete-time generalized reaction–diffusion system," *Mathematics*, vol. 12, no. 23, pp. 1–12, 2024, <https://doi.org/10.3390/math12233679>.
- [47] I. M. Batiha *et al.*, "Finite-time dynamics of the fractional-order epidemic model: stability, synchronization, and simulations," *Chaos, Solitons & Fractals: X*, vol. 13, p. 100118, 2024, <https://doi.org/10.1016/j.csf.2024.100118>.
- [48] K. -N. Wu and H. -Y. Yu, "Finite-time stabilization of stochastic delay reaction–diffusion systems," *2016 35th Chinese Control Conference (CCC)*, pp. 1713–1716, 2016, <https://doi.org/10.1109/ChiCC.2016.7553339>.
- [49] X. X. Han, K. N. Wu, and X. Ding, "Finite-time stabilization for stochastic reaction–diffusion systems with Markovian switching via boundary control," *Applied Mathematics and Computation*, vol. 385, p. 125422, 2020, <https://doi.org/10.1016/j.amc.2020.125422>.

- [50] I. M. Batiha, I. Bendib, A. Ouannas, I. H. Jebril, S. Alkhazaleh, and S. Momani, "On new results of stability and synchronization in finite time for FitzHugh–Nagumo model using Gronwall inequality and Lyapunov function," *Journal of Robotics and Control (JRC)*, vol. 5, no. 6, pp. 1897–1909, 2024, <https://doi.org/10.18196/jrc.v5i6.23211>.
- [51] O. A. Almatroud, A. Hioual, A. Ouannas, and I. M. Batiha, "Asymptotic stability results of generalized discrete-time reaction–diffusion system applied to Lengyel–Epstein and Degn–Harrison models," *Computers & Mathematics with Applications*, vol. 170, pp. 25–32, 2024, <https://doi.org/10.1016/j.camwa.2024.06.028>.
- [52] I. M. Batiha, O. Ogilat, A. Hioual, A. Ouannas, N. Anakira, A. A. Amourah, and S. Momani, "On discrete FitzHugh–Nagumo reaction–diffusion model: stability and simulations," *Partial Differential Equations in Applied Mathematics*, vol. 11, p. 100870, 2024, <https://doi.org/10.1016/j.padiff.2024.100870>.
- [53] I. Bendib, I. Batiha, A. Hioual, N. Anakira, M. Dalah, and A. Ouannas, "On a new version of Gierer–Meinhardt model using fractional discrete calculus," *Results in Nonlinear Analysis*, vol. 7, no. 2, pp. 1–15, 2024, <https://nonlinear-analysis.com/index.php/pub/article/view/345/212>.
- [54] O. Alsayyed *et al.*, "On stability of a fractional discrete reaction–diffusion epidemic model," *Fractal and Fractional*, vol. 7, no. 10, p. 729, 2023, <https://doi.org/10.3390/fractalfract7100729>.
- [55] O. A. Almatroud, A. Hioual, A. Ouannas, and G. Grassi, "On fractional-order discrete-time reaction–diffusion systems," *Mathematics*, vol. 11, no. 11, pp. 2447, 2023, <https://doi.org/10.3390/math11112447>.
- [56] K. N. Wu, H. X. Sun, P. Shi, and C. C. Lim, "Finite-time boundary stabilization of reaction–diffusion systems," *International Journal of Robust and Nonlinear Control*, vol. 28, no. 5, pp. 1641–1652, 2018, <https://doi.org/10.1002/rnc.3977>.
- [57] Y. Li, L. Hu, and Y. Ren, "Finite-time stability analysis of switched stochastic reaction–diffusion systems," *International Journal of Control*, vol. 96, no. 10, pp. 2471–2476, 2023, <https://doi.org/10.1080/00207179.2022.2097959>.
- [58] S. Momani, I. M. Batiha, I. Bendib, A. Al-Nana, A. Ouannas, and M. Dalah, "On finite-time stability of some COVID-19 models using fractional discrete calculus," *Computer Methods and Programs in Biomedicine Update*, vol. 7, p. 100188, 2025, <https://doi.org/10.1016/j.cmpbup.2025.100188>.
- [59] I. Batiha, I. Bendib, A. Ouannas, P. Agarwal, N. Anakira, I. H. Jebril, and S. Momani, "Finite-time synchronization in a novel discrete fractional SIR model for COVID-19," *Journal of Applied Mathematics and Computing*, vol. 43, no. 1, pp. 1–18, 2025, <https://doi.org/10.5269/bspm.75295>.
- [60] S. Momani, I. M. Batiha, M. S. Hijazi, I. Bendib, A. Ouannas, and N. Anakira, "Fractional-order SEIR model for COVID-19: finite-time stability analysis and numerical validation," *International Journal of Neutrosophic Science*, vol. 26, no. 1, pp. 1–10, 2025, <https://doi.org/10.54216/IJNS.260123>.
- [61] T. Abdeljawad, "On Riemann and Caputo fractional differences," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1602–1611, 2011, <https://doi.org/10.1016/j.camwa.2011.03.036>.
- [62] A. Pratap *et al.*, "Stability of discrete-time fractional-order time-delayed neural networks in complex field," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 1, pp. 419–440, 2021, <https://doi.org/10.1002/mma.6745>.
- [63] T. Hamadneh, A. Hioual, O. Alsayyed, Y. A. Al-Khassawneh, A. Al-Husban, and A. Ouannas, "Finite-time stability results for neural networks described by variable-order fractional difference equations," *Fractal and Fractional*, vol. 7, no. 8, p. 616, 2023, <https://doi.org/10.3390/fractalfract7080616>.
- [64] T. Abdeljawad, "On delta and nabla Caputo fractional differences and dual identities," *Discrete Dynamics in Nature and Society*, vol. 2013, p. 406910, 2013, <https://doi.org/10.1155/2013/406910>.
- [65] R. J. Zhang, L. Wang, and K. N. Wu, "Finite-time boundary stabilization of fractional reaction–diffusion systems," *Mathematical Methods in the Applied Sciences*, vol. 46, no. 4, pp. 4612–4627, 2023, <https://doi.org/10.1002/mma.8786>.
- [66] F. Wang, D. Chen, X. Zhang, and Y. Wu, "Finite-time stability of a class of nonlinear fractional-order system with the discrete time delay," *International Journal of Systems Science*, vol. 48, no. 5, pp. 984–993, 2017, <https://doi.org/10.1080/00207721.2016.1226985>.
- [67] Y. Zhao, Y. Wang and Z. Liu, "Finite time stability analysis for nonlinear fractional order differential systems," *Proceedings of the 32nd Chinese Control Conference*, pp. 487–492, 2013, <https://ieeexplore.ieee.org/abstract/document/6639480>.

-
- [68] H. Al-Taani, M. Abu Hammad, O. Alomari, I. Bendib, and A. Ouannas, "Finite-time control of the discrete Sel'kov–Schnakenberg model: synchronization and simulations," *AIP Advances*, vol. 15, p. 025325, 2025, <https://doi.org/10.1063/5.0257304>.
- [69] H. Jahanshahi, S. S. Sajjadi, S. Bekiros, and A. A. Aly, "On the development of variable-order fractional hyperchaotic economic system with a nonlinear model predictive controller," *Chaos, Solitons & Fractals*, vol. 144, p. 110698, 2021, <https://doi.org/10.1016/j.chaos.2021.110698>.
- [70] R. Hatamleh, I. Bendib, A. Qazza, R. Saadeh, A. Ouannas, and M. Dalah, "Finite-time stability and synchronization of the glycolysis reaction–diffusion model," *International Journal of Neutrosophic Science*, vol. 25, no. 4, pp. 371–386, 2025, <https://doi.org/10.54216/IJNS.250431>.