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Fractional ODEs and PDEs in the Range (4,5): Theoretical Analysis and Computational Aspects

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ABSTRACT

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This paper investigates two nonlinear fractional differential problems where the order of differentiation lies between four and five. The first part focuses on a class of fractional differential equations involving Caputo derivatives of order $\alpha \in (4, 5]$. Using the Banach contraction principle, we establish the existence and uniqueness of solutions and subsequently analyze their Ulam-Hyers stability. Illustrative examples are provided to show the applicability of the two main results. The second part explores the impact of fractional derivatives in wave dynamics by studying traveling wave solutions for the time and space conformable fractional Kawahara equation, where the order is less than five. The tanh method is applied to derive wave solutions, and graphical representations illustrate their behavior. Although treated separately, these two problems share a common feature: both involve fractional differential equations of intermediate order. This connection underscores the significance of such derivatives in both theoretical analysis and applied mathematical modeling.

1. Introduction

Fractional differential equations (FDEs) have emerged as powerful modeling tools across diverse scientific disciplines, especially in chemistry, physics, and dynamical systems. Their superior accuracy over classical integer-order equations stems from the additional degrees of freedom they provide [5], [9], [10], [18]. This enhanced modeling capability has spurred extensive research into solution existence and uniqueness for various FDE classes, given their critical role in describing complex physical and biological phenomena [6]–[8], [20], [26], [34], [42]. Notably, stability problems have received considerable attention, with numerous studies examining stability properties across different FDE formulations [19], [22], [23], [36]. The Caputo fractional derivative revolutionized the field by enabling the solution of viscoelasticity boundary value problems. Its key advantage lies in maintaining consistency with integer-order differential equations' initial and boundary conditions, making it particularly valuable for real-world applications.

In recent studies, various generalizations of fractional derivatives have been proposed, including the ψ-Caputo and q-Caputo forms [2], [30], [45], to improve modeling of memory effects and

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anomalous dynamics in real-world systems [3], [49]. These new operators have been utilized effectively to solve fractional Maxwell's equations [4], fractional Schrödinger systems [38], and diffusion-reaction equations in complex domains [24], [43]. Additionally, symbolic computation and Laplace-based iterative schemes have enhanced analytical techniques for solving conformable and Caputo-type FDEs [13], [31], [33], [51]. In a very recent paper, I. Jebril et al. [27] studied the following problem fractional problem under the order $\alpha \in (4,5]$:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}u(t) - a(t)f(t,u(t)) - b(t)g(t,u'(t)) - J_{0^+}^{\beta}h(t,u'(t),u''(t),u'''(t)) = 0, \ t \in [0,1], \\ u(0) = 0, \quad u'''(0) = u_3, \\ u'(0) = u_1, \quad u(1) = kJ_{0^+}^{\beta}u(\eta), \ \ \eta \in (0,1), \ k \in \mathbb{R}, \\ u''(0) = u_2, \end{cases}$$

Where: ${}^{C}D_{0^+}^{\alpha}$ denotes derivative of Caputo, with $\alpha \in (4,5]$, $J_{0^+}^{\beta}$ is the Riemann-Liouville integral of order $\beta > 0$. The authors proved new theorems for the existence and uniqueness, and Ulam-Hyers stability of solutions. The authors also gave a variety of numerical examples with discussions of their corresponding graphs. Our work addresses two fundamental aspects of high-order nonlinear differential equations:

In the first part, we investigate the question of existence and uniqueness of solutions and the Ulam Hyers stability for the the following Caputo fractional boundary value problem:

$$\begin{cases} D^{\alpha}\zeta(t) = F(t,\zeta(t),\zeta'(t),\zeta''(t),\zeta^{(3)}(t),\zeta^{(4)}(t)), \quad t \in [0,1], \\ \quad \zeta(0) = \zeta'(0) = \zeta''(0) = 0, \\ \zeta^{(3)}(1) = b_3, \quad \zeta^{(4)}(1) = b_4, \quad b_3, b_4 \in \mathbb{R}, \end{cases} \tag{1}$$

Where: ${}^{C}D^{\alpha}$ denotes the Caputo derivative of order $\alpha \in (4,5]$, $\zeta^{(i)}(t)$ represents the i-th derivative ($i = 0, \ldots, 4$), $J = [0,1]$ and $\zeta : [0,1] \to \mathbb{R}$ is the unknown function.

We are interested in this problem because many fifth-order ODEs can be viewed as limiting cases of our fractional differential problem. On the other hand, fifth-order ODEs are widely used in modeling oscillators with advanced feedback loops, electrical circuits, and nonlinear systems in engineering [15]–[17], [44].

In the second part, we employ the Tanh method to obtain new traveling wave solutions for the Kawahara equation:

$$\daleth_t^{\alpha}\chi + pf(\chi)\daleth_x\chi + \daleth_x^{\beta}(a\chi + b\daleth_x^{2\beta}\chi + c\daleth_x^{4\beta}\chi) = 0, \tag{2}$$

Where $\daleth_x^{\beta}$, $\daleth_t^{\alpha}$ denote conformable fractional derivatives with $0 < \alpha, \beta \leq 1, 5\beta \in]4,5]$.
The study of traveling wave solutions remains crucial across numerous scientific applications. While various methods exist for solving partial FDEs—including the exp-function [21], [39], (G'/G) [50], and tanh methods [41]—the tanh method stands out for its effectiveness in deriving exact solutions to nonlinear problems. Originally developed by Malfliet [35] and later enhanced by Wazwaz [48], this approach provides a robust framework for obtaining traveling wave solutions.

Our methodology draws inspiration from recent advances in the paper of M. Rakah et al. [40] where the authors studied traveling waves for evolutionary problems of the form:

$$\partial_t^{2\alpha}u + \partial_x^{\beta}(F(u)\partial_x^{3\beta}u) + \partial_x^{\beta}(G(u)\partial_x^{\beta}u) = H(u),$$

Where ∂_x^{β}, ∂_t^{α} are conformable fractional derivatives.

It has been inspired also from the work of Z. Dahmani et al. [12] where the authors developed an $(n+1)$-dimensional extended tanh method for nonlinear FDEs, demonstrating its efficacy through solutions of the fractional Burgers equation.

2. Preliminaries

We need to introduce the Caputo derivatives. For more details, we refer to the reference [32], [37].

Definition 1 Let $\alpha > 0$ and $f : J \longmapsto \mathbb{R}$ be a continuous function. The Riemann-Liouville integral is defined by:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

+

Definition 2 For any $f \in C^n(J, \mathbb{R})$ and $n-1 < \alpha \leq n$, the Caputo derivative is defined by:

$$\begin{aligned} D^{\alpha} f(t) &= I^{n-\alpha} \frac{d^n}{dt^n}(f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds. \end{aligned}$$

We reall the following results [37]:

Lemma 1 Let $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$. Then, the general solution of $D^{\alpha}\zeta(t) = 0; t \in J$ is:

$$\zeta(t) = \sum_{i=0}^{n-1} s_i t^i,$$

Where $s_i \in \mathbb{R}$, $i = 0, 1, 2, .., n-1$.

Lemma 2 If $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$, then, we have

$$I^{\alpha} D^{\alpha} \zeta(t) = \zeta(t) + \sum_{i=0}^{n-1} s_i t^i,$$

and $c_i \in \mathbb{R}$, $i = 0, 1, 2, .., n-1$.

Now, we can prove the following lemma.

Lemma 3 Let $\Psi \in C(J)$. The problem

$$\begin{cases} D^{\alpha}\zeta(t) = \Psi(t), \ \ t \in J, \\ \zeta(0) = \zeta'(0) = \zeta''(0) = 0, \quad , \\ \zeta^{(3)}(1) = b_3, \quad \zeta^{(4)}(1) = b_4, \quad b_3, \ b_4 \in \mathbb{R}, \end{cases}$$

if and only if

$$\zeta(t) = I^{\alpha}\Psi(t) + \frac{1}{6}\Big[(b_3 - b_4) + I^{\alpha-4}\Psi(1) - I^{\alpha-3}\Psi(1)\Big] t^3 + \frac{1}{24}\Big[b_4 - I^{\alpha-4}\Psi(1)\Big] t^4. \tag{3}$$

Proof: We can use Lemma 2 to observe that

$$\begin{array}{l} \zeta(t)(t) = I^{\alpha}\Psi(t) + s_0 + s_1 t + s_2 t^2 + s_3 t^3 + s_4 t^4, \\ \zeta'(t)(t) = I^{\alpha-1}\Psi(t) + s_1 + 2s_2 t + 3s_3 t^2 + 4s_4 t^3, \\ \zeta''(t)(t) = I^{\alpha-2}\Psi(t) + 2s_2 + 6s_3 t + 12 s_4 t^2, \\ \zeta^{(3)}(t)(t) = I^{\alpha-3}\Psi(t) + 6s_3 + 24 s_4 t, \\ \zeta^{(4)}(t)(t) = I^{\alpha-4}\Psi(t) + 24 s_4, \end{array} \tag{4}$$

The conditions allows us to get

$$\zeta(0)=\zeta'(0)=\zeta''(0)=0 \Rightarrow s_0=s_1=s_2=0,$$

Then

$$\zeta^{(3)}(1)=b_3 \Rightarrow I^{\alpha-3}\Psi(1)+6s_3+24s_4=b_3,$$
$$\zeta^{(4)}(1)=b_4 \Rightarrow I^{\alpha-4}\Psi(1)+24s_4=b_4,$$

we find the values

$$s_4=\frac{b_4}{24}-\frac{1}{24}I^{\alpha-4}\Psi(1),$$

$$s_3=\frac{1}{6}\left[(b_3-b_4)+I^{\alpha-4}\Psi(1)-I^{\alpha-3}\Psi(1)\right].$$

We end the proof of our lemma. In what follows, we consider the space:

$$\Xi:=\{\zeta\in C^4(J,\mathbb{R})\},$$

and its norm

$$\|\zeta\|_\Xi=Max\{\|\zeta\|_\infty\,,\|\zeta'\|_\infty\,\|\zeta''\|_\infty\,\|\zeta^{(3)}\|_\infty\,\|\zeta^{(4)}\|_\infty\},$$

where,

$$\|\zeta\|_\infty=\sup_{t\in J}|\zeta(t)|.$$

Then, we will consider the nonlinear operator $\Upsilon:\Xi\to\Xi$ defined by:

$$\begin{aligned}\Upsilon\zeta &= I^\alpha F(t,\zeta(t),\zeta'(t)+\zeta''(t)+\zeta^{(3)}(t)+\zeta^{(4)}(t))\\ &+ \frac{1}{6}\Big[(b_3-b_4)+I^{\alpha-4}F(1,\zeta(1),\zeta'(1)+\zeta''(1)+\zeta^{(3)}(1)+\zeta^{(4)}(1))\\ &- I^{\alpha-3}F(1,\zeta(1),\zeta'(1)+\zeta''(1)+\zeta^{(3)}(1)+\zeta^{(4)}(1))\Big]t^3\\ &+ \frac{1}{24}\Big[b_4-I^{\alpha-4}F(1,\zeta(1),\zeta'(1)+\zeta''(1)+\zeta^{(3)}(1)+\zeta^{(4)}(1))\Big]t^4.\end{aligned}$$

At this step, we are able to use fixed point theory to study the first part of our present work.

3. Main Results

We consider the following hypotheses:

$(L1)$: The function F defined on $J\times\mathbb{R}^5$ is continuous.

$(L2)$: There exist continuous functions h_i, $(i=\overline{1,5})$, such that for any t $\in J$, $\zeta_i,\zeta_i^*\in\mathbb{R}$, $(i=\overline{1,5})$,

$$|F(t,\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5)-F(t,\zeta_1^*,\zeta_2^*,\zeta_3^*,\zeta_4^*,\zeta_5^*)\le\sum_{i=1}^{5}h_i(t)|\zeta_i-\zeta_i{}^*|,$$

We consider:

$$\begin{array}{lll} k_1=\sup\limits_{t\in J}|h_1(t)|, & k_2=\sup\limits_{t\in J}|h_2(t)|, & k_3=\sup\limits_{t\in J}|h_3(t)|,\\ k_4=\sup\limits_{t\in J}|h_4(t)|, & k_5=\sup\limits_{t\in J}|h_5(t)|, & ,K^*=Max\{k_1,k_2,k_3,k_4,k_5\}.\end{array}$$

Also, we need to consider the quantities:

$$\Phi=|b_3-b4|+\frac{K^*}{\Gamma(\alpha-3)}+\frac{K^*}{\Gamma(\alpha-4)},$$

$$\Theta = |b_4| + \frac{K^*}{\Gamma(\alpha-3)}.$$

$$X_i = \frac{K^*}{\Gamma(\alpha+1-i)},$$

$$X^* = Max\{X_i, i = \overline{0,4}\}.$$

3.1. A Unique Solution

We prove the following theorem.

Theorem 1 If $(L_1) - (L_2)$ are valid, and

$$X^* + \Phi + \Theta \in \,]0,1[\,,$$

then, problem (1) has a unique solution.

Proof: We need to prove that Υ is contractive. Let $(\zeta, \zeta^*) \in \Xi^2$, we can write

$$\begin{aligned} \|\Upsilon\zeta - \Upsilon\zeta^*\|_\infty \;&\leq\; \Bigg[\frac{K^*}{\Gamma(\alpha+1)} + \frac{1}{6}\bigg(|b_3 - b4| + \frac{K^*}{\Gamma(\alpha-3)} + \frac{K^*}{\Gamma(\alpha-2)}\bigg) \\ &\quad + \frac{1}{24}\bigg(|b_4| + \frac{K^*}{\Gamma(\alpha-3)}\bigg)\Bigg] \|\zeta - \zeta^{'}\|_\Xi \\ &\leq\; \Bigg[\frac{K^*}{\Gamma(\alpha+1)} + \frac{1}{6}\Phi + \frac{1}{24}\Theta\Bigg] \|\zeta - \zeta^{'}\|_\Xi. \end{aligned}$$

We have

$$\begin{aligned} \Upsilon'\zeta \;&=\; I^{\alpha-1}F(t, \zeta(t), \zeta'(t) + \zeta''(t) + \zeta^{(3)}(t) + \zeta^{(4)}(t)) \\ &+\; \frac{1}{2}\Big[(b_3 - b_4) + I^{\alpha-4}F(1, \zeta(1), \zeta'(1) + \zeta''(1) + \zeta^{(3)}(1) + \zeta^{(4)}(1)) \\ &-\; I^{\alpha-3}F(1, \zeta(1), \zeta'(1) + \zeta''(1) + \zeta^{(3)}(1) + \zeta^{(4)}(1))\Big] t^2 \\ &+\; \frac{1}{6}\Big[b_4 - I^{\alpha-4}F(1, \zeta(1), \zeta'(1) + \zeta''(1) + \zeta^{(3)}(1) + \zeta^{(4)}(1))\Big] t^3, \end{aligned}$$

$$\begin{aligned} \Upsilon''\zeta \;&=\; I^{\alpha-2}F(t, \zeta(t), \zeta'(t) + \zeta''(t) + \zeta^{(3)}(t) + \zeta^{(4)}(t)) \\ &+\; \Big[(b_3 - b_4) + I^{\alpha-4}F(1, \zeta(1), \zeta'(1) + \zeta''(1) + \zeta^{(3)}(1) + \zeta^{(4)}(1)) \\ &-\; I^{\alpha-3}F(1, \zeta(1), \zeta'(1) + \zeta''(1) + \zeta^{(3)}(1) + \zeta^{(4)}(1))\Big] t \\ &+\; \frac{1}{2}\Big[b_4 - I^{\alpha-4}F(1, \zeta(1), \zeta'(1) + \zeta''(1) + \zeta^{(3)}(1) + \zeta^{(4)}(1))\Big] t^2, \end{aligned}$$

$$\begin{aligned} \Upsilon^{(3)}\zeta \;&=\; I^{\alpha-3}F(t, \zeta(t), \zeta'(t) + \zeta''(t) + \zeta^{(3)}(t) + \zeta^{(4)}(t)) \\ &+\; (b_3 - b_4) + I^{\alpha-4}F(1, \zeta(1), \zeta'(1) + \zeta''(1) + \zeta^{(3)}(1) + \zeta^{(4)}(1)) \\ &-\; I^{\alpha-3}F(1, \zeta(1), \zeta'(1) + \zeta''(1) + \zeta^{(3)}(1) + \zeta^{(4)}(1)) \\ &+\; \Big[b_4 - I^{\alpha-4}F(1, \zeta(1), \zeta'(1) + \zeta''(1) + \zeta^{(3)}(1) + \zeta^{(4)}(1))\Big] t, \end{aligned}$$

$$\begin{aligned} \Upsilon^{(4)}\zeta \;&=\; I^{\alpha-4}F(t, \zeta(t), \zeta'(t) + \zeta''(t) + \zeta^{(3)}(t) + \zeta^{(4)}(t)) \\ &+\; b_4 - I^{\alpha-4}F(1, \zeta(1), \zeta'(1) + \zeta''(1) + \zeta^{(3)}(1) + \zeta^{(4)}(1)). \end{aligned}$$

On the other hand,

$$\begin{aligned}
\|\Upsilon'\zeta-\Upsilon'\zeta^{*}\|_{\infty} &\leq \left[\frac{K^{*}}{\Gamma(\alpha)}+\frac{1}{2}\left(|b_3-b4|+\frac{K^{*}}{\Gamma(\alpha-3)}+\frac{K^{*}}{\Gamma(\alpha-4)}\right)\right.\\
&\quad\left.+\frac{1}{2}\left(|b_4|+\frac{K^{*}}{\Gamma(\alpha-3)}\right)\right]\|\zeta-\zeta'\|_{\Xi}\\
&\leq \left[\frac{K^{*}}{\Gamma(\alpha)}+\frac{1}{2}\Phi+\frac{1}{6}\Theta\right]\|\zeta-\zeta'\|_{\Xi},
\end{aligned}$$

$$\begin{aligned}
\|\Upsilon''\zeta-\Upsilon''\zeta^{*}\|_{\infty} &\leq \left[\frac{K^{*}}{\Gamma(\alpha-1)}+|b_3-b4|+\frac{K^{*}}{\Gamma(\alpha-3)}+\frac{K^{*}}{\Gamma(\alpha-4)}\right.\\
&\quad\left.+\frac{1}{2}\left(|b_4|+\frac{K^{*}}{\Gamma(\alpha-3)}\right)\right]\|\zeta-\zeta'\|_{\Xi}\\
&\leq \left[\frac{K^{*}}{\Gamma(\alpha-1)}+\Phi+\frac{1}{2}\Theta\right]\|\zeta-\zeta'\|_{\Xi},
\end{aligned}$$

$$\begin{aligned}
\|\Upsilon^{(3)}\zeta-\Upsilon^{(3)}\zeta^{*}\|_{\infty} &\leq \left[\frac{K^{*}}{\Gamma(\alpha-2)}+|b_3-b4|+\frac{K^{*}}{\Gamma(\alpha-3)}+\frac{K^{*}}{\Gamma(\alpha-4)}\right.\\
&\quad\left.+|b_4|+\frac{K^{*}}{\Gamma(\alpha-3)}\right]\|\zeta-\zeta'\|_{\Xi}\\
&\leq \left[\frac{K^{*}}{\Gamma(\alpha-2)}+\Phi+\Theta\right]\|\zeta-\zeta'\|_{\Xi},
\end{aligned}$$

$$\begin{aligned}
\|\Upsilon^{(4)}\zeta-\Upsilon^{(4)}\zeta^{*}\|_{\infty} &\leq \left[\frac{2K^{*}}{\Gamma(\alpha-3)}+|b4|\right]\|\zeta-\zeta'\|_{\Xi}\\
&\leq \left[\frac{K^{*}}{\Gamma(\alpha-3)}+\Phi+\Theta\right]\|\zeta-\zeta'\|_{\Xi}.
\end{aligned}$$

Consequently,

$$\|\Upsilon\zeta-\Upsilon\zeta^{*}\|_{\Xi} \leq \left[X^{*}+\Phi+\Theta\right]\|\zeta(t)-\zeta^{*}(t)\|_{\Xi}.$$

By the Banach contraction principle, Υ has a unique fixed point which is the unique solution of (1).

3.2. Stability in the Sense of Ulam-Hyers

Definition 3 The equation (1) has the Ulam Hyers stability if there exists a real number $\eta>0$, such that for each $\varpi>0, t\in J$ and for each $\zeta\in\Xi$ solution of the inequality

$$\left|D^{\alpha}\zeta(t)-F(t,\zeta(t),\zeta'(t),\zeta''(t),\zeta^{(3)}(t),\zeta^{(4)}(t))\right|\leq\varpi, \tag{5}$$

under the following conditions:

$$\begin{cases}\zeta(0)=\zeta'(0)=\zeta''(0)=0, & ,\\ \zeta^{(3)}(1)=b_3, \quad \zeta^{(4)}(1)=b_4, & b_3,\ b_4\in\mathbb{R},\end{cases}$$

There exists $\zeta^* \in \Xi$ a solution of (1), such that

$$\|\zeta - \zeta^*\|_\Xi \leq \eta\varpi.$$

Definition 4 The equation (1) has the Ulam Hyers stability in the generalized sense if there exists $\eta \in C(\mathbb{R}^+, \mathbb{R}^+)$; $\eta(0) = 0$, such that for each $\varpi > 0$, and for any $\zeta \in \Xi$ solution of (5), there exists a solution $\zeta^* \in \Xi$ of (1)

$$\|\zeta - \zeta^*\|_\Xi < \eta(\varpi).$$

We prove the second main result.

Theorem 2 Under the conditions of Theorem (1), problem (1) is Ulam Hyers stable.

Proof: Let $\zeta \in \Xi$ be a solution of (5), and let, by Theorem 1, $\zeta^* \in \Xi$ be the unique solution of (1). By integration of (5), we can obtain

$$\begin{aligned}
&\Big|\zeta(t) - I^\alpha F(t, \zeta(t), \zeta'(t) + \zeta''(t) + \zeta^{(3)}(t) + \zeta^{(4)}(t))\\
&-\frac{1}{6}\Big[(b_3 - b_4) + I^{\alpha-4}F(1, \zeta(1), \zeta'(1) + \zeta''(1) + \zeta^{(3)}(1) + \zeta^{(4)}(1))\\
&+I^{\alpha-3}F(1, \zeta(1), \zeta'(1) + \zeta''(1) + \zeta^{(3)}(1) + \zeta^{(4)}(1))\Big]t^3\\
&-\frac{1}{24}\Big[b_4 - I^{\alpha-4}F(1, \zeta(1), \zeta'(1) + \zeta''(1) + \zeta^{(3)}(1) + \zeta^{(4)}(1))\Big]t^4\Big| \leq \frac{\varpi}{\Gamma(\alpha+1)}\\
&.
\end{aligned} \tag{6}$$

Using (5) and (6), we observe that

$$\|\zeta - \zeta^*\|_\Xi \leq \frac{\varpi}{\Gamma(\alpha+1)} + \Big[X^* + \Phi + \Theta\Big]\|\zeta - \zeta^*\|_\Xi. \tag{7}$$

Hence,

$$\|\zeta - \zeta^*\|_\Xi \quad \leq \quad \frac{\varpi}{\Gamma(\alpha+1)\Big(1 - X^* - \Phi - \Theta\Big)} \leq \eta\varpi.$$

Thus,

$$\|\zeta - \zeta^*\|_\Xi \quad \leq \quad \eta\varpi.$$

Where

$$\eta = \frac{1}{\Gamma(\alpha+1)\Big(1 - X^* - \Phi - \Theta\Big)}.$$

Therefore, (1) has the Ulam Hyers stability.

Remark 1 In the case $\eta(\varpi) = \varpi.\eta$, we obtain the generalized Ulam Hyers stability for (1).

4. Two Examples

Example 1 Consider the following first example:

$$\begin{cases}
D^{4.6}\zeta(t) = \dfrac{cos\Big(\zeta(t) + \zeta'(t) + \zeta''(t) + \zeta^{(3)}(t) + \zeta^{(4)}(t)\Big)}{100(t^2+1)}, \quad t \in J,\\
\zeta(0) = \zeta'(0) = \zeta''(0) = 0, \quad ,\\
\zeta^{(3)}(1) = \dfrac{1}{3}, \quad \zeta^{(4)}(1) = \dfrac{1}{4},
\end{cases} \tag{8}$$

We have $\Phi = 0.1012$, $\Theta = 0.2612$, $X^* = 0.0112$, and then, $X^* + \Phi + \Theta = 0.3736 \in]0, 1[$. Therefore, problem (8) has a unique solution, by Theorem 1. It also has the Ulam-Hyers stable.

Example 2 Consider the following second example:

$$\begin{cases} D^{4.8}\zeta(t) = \dfrac{1}{50e^{t^2+1}}\left(\zeta(t) + \zeta'(t) + sin(\zeta''(t) + \zeta^{(3)}(t)) + \dfrac{|\zeta^{(4)(t)}|}{|\zeta^{(4)}(t)|+1}\right), \; t \in J, \\ \zeta(0) = \zeta'(0) = \zeta''(0) = 0, \\ \zeta^{(3)}(1) = \dfrac{1}{8}, \quad \zeta^{(4)}(1) = e^{-2}. \end{cases} \tag{9}$$

We have $\Phi = 0.0156$, $\Theta = 0.1279$, $X^* = 0.0029$. Then $X^* + \Phi + \Theta = 0.1464 \in]0,1[$. Thus, problem (9) has a unique solution by Theorem 1. Also, thanks to Theorem 1, this problem is Ulam-Hyers stable.

5. Conformable Fractional Kawahara Equation

Let us consider the following problem:

$$\urcorner_t^{\alpha}\chi + pf(\chi)\urcorner_x\chi + \urcorner_x^{\beta}(a\chi + b\urcorner_x^{2\beta}\chi + c\urcorner_x^{4\beta}\chi) = 0, \tag{10}$$

Where, $\urcorner_x^{\beta}, \urcorner_t^{\alpha}$ are the conformable fractional derivative, with $0 < \alpha, \beta \leq 1$.

It is to note that: When $\alpha = \beta = b = -c = p = 1$, and $f(\chi) = \chi, a = 0$ the above problem is transformed into the fifth-order Kawahara equation:

$$\chi_t + \chi\chi_x + \chi_{xxx} - \chi_{xxxxx} = 0. \tag{11}$$

When $\alpha = \beta = a = p = 1$, and $f(\chi) = \chi^2$ the above conformable problem is transformed into the Modified Kawahara equation:

$$\chi_t + \chi^2\chi_x + \chi_x + b\chi_{xxx} + c\chi_{xxxxx} = 0. \tag{12}$$

To study the above conformable problem, we need to recall the following conformable notions, see [1], [11], [12], [29]:

5.1. Conformable Fractional Derivatives

Definition 5 Let $f : (0, \infty) \to \mathbb{R}$. Then, the conformable fractional derivative of order α is defined by

$$(\urcorner^{\alpha} f)(t) = \frac{\partial^{\alpha} h(t,x)}{\partial t^{\alpha}} = \lim_{\varepsilon \to 0}\left(\frac{h\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}\right), \quad t > 0, \;\; 0 < \alpha \leq 1.$$

When $\alpha = 1$, the above formula gives the well known standard derivative.

Definition 6 The conformable fractional integral of a function $h : (0, \infty) \to \mathbb{R}$ of order α is defined as

$$(I^{\alpha} f)(t) = \int_0^t \tau^{\alpha-1} h(\tau)\, d\tau, \qquad 0 < \alpha \leq 1.$$

The following properties are important for us.

$$I^{\alpha}\urcorner^{\alpha} h(t) = h(t) - h(0),$$

and

$$(\urcorner^{\alpha} f)(t) = t^{1-\alpha}\frac{dh(t)}{dt}.$$

5.1.1. Tanh Method

Let us first consider the general case of the equation [12]:

$$\mathcal{F}\left(\chi, \daleth_t^{\alpha}\chi, \daleth_x^{\beta}\chi, \daleth_t^{2\alpha}\chi, \daleth_t^{\alpha}(\daleth_x^{\beta}\chi), \daleth_x^{2\beta}\chi, ...\right) = 0, \tag{13}$$

Where $\daleth_t^{\alpha}\chi$ is the conformable fractional derivative of χ of order $\alpha, 0 < \alpha \leq 1$. Then, let us introduce

$$\varphi = \tfrac{k}{\alpha}t^{\alpha} + \tfrac{\omega}{\beta}x^{\beta}, \tag{14}$$

Where k and ω are constants. So, (13) can be transformed to:

$$\mathcal{L}\left(Z, Z', Z'', Z''', ...\right) = 0, \tag{15}$$

We then introduce the variable

$$\aleph = tanh(\xi), \tag{16}$$

so, we get

$$\begin{aligned}
\tfrac{d}{d\varphi} &= \left(1-\aleph^2\right)\tfrac{d}{d\aleph}, \\
\tfrac{d^2}{d\varphi^2} &= -2\aleph\left(1-\aleph^2\right)\tfrac{d}{d\aleph} + \left(1-\aleph^2\right)^2\tfrac{d^2}{\aleph^2}, \\
\tfrac{d^3}{d\varphi^3} &= 2\left(1-\aleph^2\right)\left(3\aleph^2-1\right)\tfrac{d}{d\aleph} - 6\aleph\left(1-\aleph^2\right)^2\tfrac{d^2}{d\aleph^2} + \left(1-\aleph^2\right)^3\tfrac{d^3}{d\aleph^3}, \\
\tfrac{d^4}{d\varphi^4} &= -8\aleph\left(1-\aleph^2\right)\left(3\aleph^2-2\right)\tfrac{d}{d\aleph} + 4\left(1-\aleph^2\right)^2\left(9\aleph^2-2\right)\tfrac{d^2}{d\aleph^2} \\
& \quad -12\aleph\left(1-\aleph^2\right)^3\tfrac{d^3}{d\aleph^3} + \left(1-\aleph^2\right)^4\tfrac{d^4}{d\aleph^4}.
\end{aligned} \tag{17}$$

Now, we assume

$$u(x,t) = U(\varphi) = W(\aleph) = \textstyle\sum_{i=0}^{s} a_i\aleph^i, \tag{18}$$

Where s is a positive integer.

5.2. Application

We consider the problem (see [25], [28]):

$$\daleth_t^{\alpha}\chi + p\chi^2\daleth_x\chi + a\daleth_x^{\beta}\chi + b\daleth_x^{3\beta}\chi + c\daleth_x^{5\beta}\chi = 0, \tag{19}$$

We need first (14) to transform (19) into the following equation:

$$kU_{\varphi} + p\omega U^2U_{\varphi} + a\omega U_{\varphi} + b\omega^3U_{\varphi\varphi\varphi} + c\omega^5U_{\varphi\varphi\varphi\varphi\varphi} = 0. \tag{20}$$

Integrating (20) with respect to φ, we can write

$$c\omega^5U_{\varphi\varphi\varphi\varphi} + b\omega^3U_{\varphi\varphi} + (k+a\omega)U + \frac{p}{3}\omega U^3 = 0. \tag{21}$$

Substituting (17) and (18) in (21), we obtain

$$\begin{aligned}
& c\omega^5\Big[-8\aleph\left(1-\aleph^2\right)\left(3\aleph^2-2\right)\frac{dW}{d\aleph} + 4\left(1-\aleph^2\right)^2\left(9\aleph^2-2\right)\frac{d^2W}{d\aleph^2} \\
& -12\aleph\left(1-\aleph^2\right)^3\frac{d^3W}{d\aleph^3} + \left(1-\aleph^2\right)^4\frac{d^4W}{d\aleph^4}\Big] + b\omega^3\Big[-2\aleph\left(1-\aleph^2\right)\frac{dW}{d\aleph} \\
& + \left(1-\aleph^2\right)^2\frac{d^2W}{\aleph^2} + (k+a\omega)W + \frac{p}{3}\omega W^3 = 0.
\end{aligned} \tag{22}$$

Balancing ($\aleph^8\frac{d^4W}{d\aleph^4}$ vs W^3) gives

$$8 + s - 4 = 3s$$

so that $s = 2$. This implies that the solution is given by

$$W(\aleph) = a_0 + a_1\aleph + a_2\aleph^2. \tag{23}$$

Substituting (23) into (22), we can get

$$\begin{aligned} &c\omega^5\Big[-8\aleph\left(1-\aleph^2\right)\left(3\aleph^2-2\right)\left(a_1+2a_2\aleph\right)+8a_2\left(1-\aleph^2\right)^2\left(9\aleph^2-2\right)\Big] \\ &+b\omega^3\Big[-2\aleph\left(1-\aleph^2\right)\left(a_1+2a_2\aleph\right)+2a_2\left(1-\aleph^2\right)^2\Big]+(k+a\omega)(a_0+a_1\aleph+a_2\aleph^2) \\ &+\frac{p\omega}{3}(a_0+a_1\aleph+a_2\aleph^2)^3=0. \end{aligned} \tag{24}$$

Then, we have the system:

$$\begin{cases} \aleph^0 := \frac{1}{3}p\omega a_0^3 + 2b\omega^3 a_2 + ka_0 - 16c\omega^5 a_2 + a\omega a_0 = 0, \\ \aleph^1 : 16c\omega^5 a_1 - 2b\omega^3 a_1 + p\omega a_0^2 a_1 + a\omega a_1 + ka_1 = 0, \\ \aleph^2 : 136c\omega^5 a_2 - 8b\omega^3 a_2 + p\omega a_0^2 a_2 + p\omega a_0 a_1^2 + a\omega a_2 + ka_2 = 0, \\ \aleph^3 : -40c\omega^5 a_1 + 2b\omega^3 a_1 + \frac{1}{3}p\omega a_1^3 + 2p\omega a_0 a_1 a_2 = 0, \\ \aleph^4 : -240c\omega^5 a_2 + 6b\omega^3 a_2 + p\omega a_0 a_2^2 + p\omega a_1^2 a_2 = 0, \\ \aleph^5 : 24c\omega^5 a_1 + p\omega a_1 a_2^2 = 0, \\ \aleph^6 : 120c\omega^5 a_2 + \frac{1}{3}p\omega a_2^3 = 0. \end{cases} \tag{25}$$

We solve this system with the aid of Maple, and we obtain traveling wave solutions of (25) as follows shown in Fig. 1 and Fig. 2:

Case 1:

$$a_0 = 3b\sqrt{\frac{-1}{10cp}}, a_1 = 0, a_2 = \pm 3b\sqrt{\frac{-1}{10cp}}, \omega = \pm\sqrt{\frac{-b}{20c}}, k = \pm\sqrt{\frac{-b}{20c}}\left(a - \frac{4b^2}{25c}\right),$$

$$u(x,t) = 3b\sqrt{\frac{-1}{10cp}}\Big(1 \pm \tanh^2(\xi)\Big). \tag{26}$$

Case 2:

$$a_0 = -3b\sqrt{\frac{-1}{10cp}}, a_1 = 0, a_2 = \pm 3b\sqrt{\frac{-1}{10cp}}, \omega = \pm\sqrt{\frac{-b}{20c}}, k = \pm\sqrt{\frac{-b}{20c}}\left(a - \frac{4b^2}{25c}\right),$$

$$u(x,t) = 3b\sqrt{\frac{-1}{10cp}}\Big(-1 \pm \tanh^2(\xi)\Big). \tag{27}$$

Case 3:

$$a_0 = \sqrt{\frac{-(\sqrt{15}i+1)}{16cp}}\left(\frac{b(5+\sqrt{15}i)}{10}+b\right), a_1 = 0, a_2 = \pm 3\sqrt{\frac{-(\sqrt{15}b^2i+b^2)}{16cp}},,$$

$$\omega = \pm\frac{\sqrt{5bc\left(5+\sqrt{15}i\right)}}{20c}, k = \pm\frac{\sqrt{5bc\left(5+\sqrt{15}i\right)}\left(\frac{3b^2\left(5+\sqrt{15}i\right)}{4} - 20ac - b^2\right)}{400c^2},$$

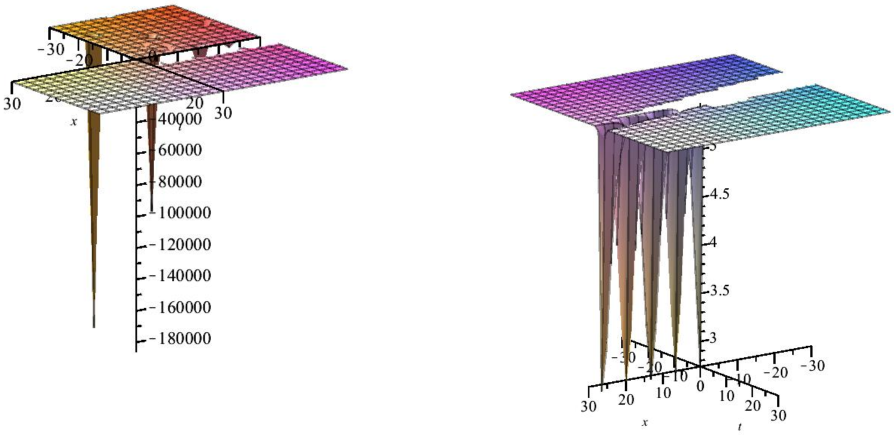
$$u(x,t) = \sqrt{\frac{-(\sqrt{15}i+1)}{16cp}}\Big(\frac{b(15+\sqrt{15}i)}{10} \pm 3b\tanh^2(\xi)\Big). \tag{28}$$

Case 4:

$$a_0 = -\sqrt{\frac{-(\sqrt{15}i+1)}{16cp}}\left(\frac{b(5+\sqrt{15}i)}{10}+b\right), a_1 = 0, a_2 = \pm 3\sqrt{\frac{-(\sqrt{15}b^2i+b^2)}{16cp}},,$$

$$\omega = \pm\frac{\sqrt{5bc\left(5+\sqrt{15}i\right)}}{20c}, k = \pm\frac{\sqrt{5bc\left(5+\sqrt{15}i\right)}\left(\frac{3b^2\left(5+\sqrt{15}i\right)}{4} - 20ac - b^2\right)}{400c^2},$$

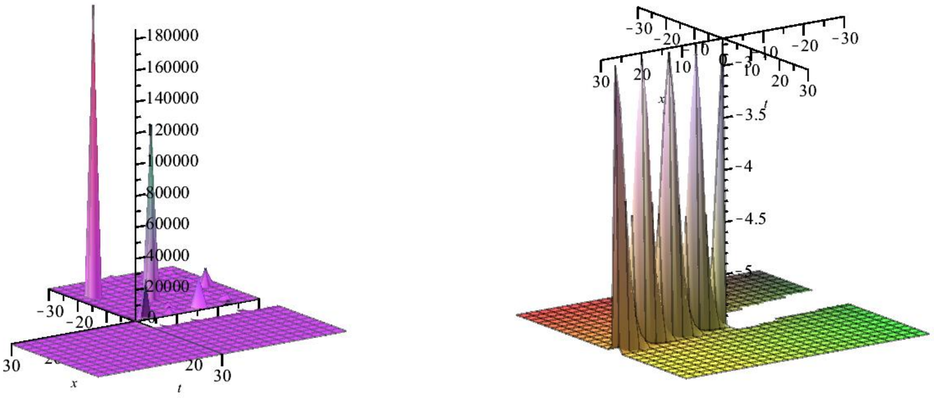
$$u(x,t) = \sqrt{\frac{-(\sqrt{15}i+1)}{16cp}}\Big(-\frac{b(15+\sqrt{15}i)}{10} \pm 3b\tanh^2(\xi)\Big). \tag{29}$$



(a) $\alpha = \frac{5}{10}, \beta = \frac{5}{10}$.

(b) $\alpha = \frac{9}{10}, \beta = \frac{9}{10}$.

Fig. 1. Traveling wave and behavior of solution (26); $-30 \leq x \leq 10$, $-30 \leq t \leq 10$



(a) $\alpha = \frac{5}{10}, \beta = \frac{5}{10}$.

(b) $\alpha = \frac{9}{10}, \beta = \frac{9}{10}$

Fig. 2. Traveling wave and behavior of solution (27); $-30 \leq x \leq 10$, $-30 \leq t \leq 10$

6. Conclusion

This study examined two distinct nonlinear fractional differential problems, both characterized by derivative orders that are between four and five. The first part established existence, uniqueness, and stability results for a class of Caputo-type fractional differential equations, reinforcing their theoretical results. The second part demonstrated the influence of fractional derivatives in wave dynamics by analyzing traveling wave solutions of the conformable fractional Kawahara equation. These problems share a common structure, showing the importance of intermediate fractional orders in both theoretical and applied contexts. This connection underscores the role of fractional calculus in diverse mathematical problems, bridging Ulam Hyers stability analysis and wave propagation phenomena.

The constructed traveling wave solutions of the conformable fractional Kawahara equation provide valuable insight into the behavior of nonlinear dispersive wave systems with memory and hereditary properties. These solutions highlight how intermediate-order conformable derivatives can effectively describe phenomena such as wave steepening, soliton-like structures, and energy dispersion in media governed by fractional dynamics. Although the tanh method successfully yielded exact analytical solutions, the current study did not explore their qualitative stability or long-term behavior under perturbations. Moreover, the lack of numerical validation leaves open questions regarding the robustness of the solutions when applied to more complex or irregular initial conditions. Therefore, a more comprehensive analysis involving numerical simulations or spectral methods would be essential to confirm the effectiveness and accuracy of the proposed analytical solutions in real-world scenarios.

Future work could aim to develop adaptive numerical solvers specifically tailored to conformable fractional PDEs, allowing researchers to simulate more general forms of the Kawahara equation with time-dependent coefficients or forcing terms. Another direction is to apply the methodology to other important nonlinear models in fractional form, such as the fractional Boussinesq, Zakharov-Kuznetsov, or KdV-Burger equations, to investigate the role of fractional orders in multidimensional systems. Additionally, future studies might examine how varying the order of the fractional derivative affects wave speed, amplitude, and stability. From an application perspective, extending the results to model physical processes in nonlinear optics, plasma physics, and fluid transport would bridge the gap between theory and practice. Interdisciplinary collaboration with experimentalists could also open the path for validating these fractional wave models in laboratory settings.

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