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Derivation of Necessary Conditions for Optimal Control of the Implicit Burgers' Equation Using the Variational Principle

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***Abstract*—Controlling fluid dynamics is a challenging problem that arises in many fields including aeronautical, biological and chemical engineering. Burgers' equation is used to describe fundamental flow phenomena. Implicit systems belong to a more generalized class of systems than a class of explicit systems, because they can additionally contain algebraic constraints. Although the optimal control problem of the explicit form of Burgers' equation has been already explored in many papers, the optimal control problem of implicit form of Burgers' equation is still an open problem as far as general classes of systems are concerned. In this paper, necessary conditions for an evaluation function to be optimized are derived on the basis of the variational principle for the optimal control problem of the implicit form of Burgers' equation.**

Keywords*— Optimal Control; Nonlinear Systems; Implicit Systems; Control Engineering; System Design

I. INTRODUCTION

Controlling fluid dynamics is a challenging problem that arises in many fields including aeronautical, biological and chemical engineering. Burgers' equation is used to describe fundamental flow phenomena. Therefore, using Burgers' equations can be regarded as a natural first step towards developing a method for controlling flows.

The boundary feedback control issue for a specific category of systems represented by Burgers' equation was explored in references [1]–[7]. In [1], a boundary control strategy utilizing linearization was introduced, which successfully achieves local stabilization of Burgers' equation. However, this strategy necessitates that the initial solution remains sufficiently small. To eliminate this limitation on the initial solution's size, the global existence and uniqueness of a solution for a Burgers' equation were demonstrated in [2]. The control techniques outlined in [2] rely on a local Lyapunov function, implying that the initial states of the system must lie within a local attractor. To address this limitation of local stability, the global exponential stability of Burgers' equation in the L^2 and H^1 norms was examined in [3] and [4], respectively. Additionally, a backstepping boundary control law that incorporates actuator dynamics was proposed for Burgers' equation in [5]. For practical applications where viscosity is not known, an adaptive control approach for Burgers' equation was devised in [6]. Furthermore, an adjoint-based optimal control method was introduced for Burgers' equation using a high-order spectral difference technique in [7].

Model predictive control (MPC) [8]–[11], also known as receding horizon control [12]–[14], is a well-established control method in which the current control input is obtained by solving a finite-horizon open-loop optimal control problem using the current state of the system as the initial state, and this procedure is repeated at each sampling instant. Thus, MPC is a type of optimal feedback control in which the control performance over a finite future is optimized with a performance index that has a moving initial time and terminal time. MPC method for the explicit form of Burgers' equation has been proposed in [15]. On the other hand, several optimal control problems of the explicit form of Burgers' equation have been studied in several papers [16]–[28].

It is worth noting that implicit systems belong to a more generalized class of systems than a class of explicit systems, because they can additionally contain algebraic constraints. It can be seen from Fig. 1 that a class of implicit systems includes a class of explicit systems. Although the aforementioned studies [1]–[7] have achieved progress in controlling the explicit form of Burgers' equation, the control problem of the implicit form of Burgers' equation has not yet been addressed. Moreover, various optimal control problems of Burgers' equation have been studied in several papers [15]–[28]. However, a class of Burgers' equation addressed in those papers is limited to a class of explicit form of Burgers' equation.

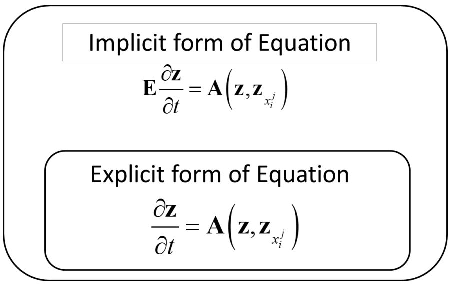

Fig. 1. Inclusion of explicit system by implicit system.

To the best of my knowledge, the optimal control problem of implicit form of Burgers' equation is still an open problem as far as general classes of systems are concerned. The objective of this study is to solve the optimal control problem of implicit form of Burgers' equation. This paper aims to establish a more generalized control system design method compared to the method proposed in [15] by solving this problem. For this purpose, necessary conditions for an

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evaluation function to be optimized are derived for optimal control problem of implicit form of Burgers' equation.

This paper is organized as follows. In section 2, we introduce some notations and the system model. In section 3, we consider the optimal control problem of Burgers' equation. In section 4, using the variational principle, we derive the stationary conditions, which must be satisfied for optimizing the performance index. Finally, we provide concluding remarks in section 5.

II. NOTATION AND SYSTEM MODEL

The transpose of matrix $\boldsymbol{A}$ is denoted by $\boldsymbol{A}'$. Let $\mathrm{diag}\{\cdots\}$ denote a diagonal matrix. Furthermore, let $\boldsymbol{z} = [z_1, \cdots, z_n]' \in \mathbb{R}^n$ and $\boldsymbol{x} = [x_1, \cdots, x_n]' \in \mathbb{R}^n$ denote the state and spatial vectors, respectively. Let $t \in \mathbb{R}$ denote the temporal variable. Without loss of generality, we restrict our attention to the range $0 \le x_i \le h$ for all $i = 1, \cdots, n$, where h is a positive constant. Let Ω and $\partial\Omega_i$ be sets defined by $\Omega := \prod_{i=1}^{n} \{x_i | 0 \le x_i \le h\}$ and $\partial\Omega_i := \{x_i | x_i = 0, x_i = h\} \cap \Omega$, respectively. Let $\boldsymbol{z}(\boldsymbol{x}, t) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n$ be a continuous vector-valued function with respect to $\boldsymbol{x}$ and t. Then, for $i = 1, \cdots, n$ and $j = 1,2$ we introduce the following notations.

$$\boldsymbol{z}_t(\boldsymbol{x}, t) := \frac{\partial \boldsymbol{z}(\boldsymbol{x}, t)}{\partial t} := \left[\frac{\partial z_1(\boldsymbol{x}, t)}{\partial t}, \cdots, \frac{\partial z_n(\boldsymbol{x}, t)}{\partial t}\right]',$$

$$\boldsymbol{z}_{x_i^j}(\boldsymbol{x}, t) := \frac{\partial^j \boldsymbol{z}(\boldsymbol{x}, t)}{\partial x_i^j} := \left[\frac{\partial^j z_1(\boldsymbol{x}, t)}{\partial x_i^j}, \cdots, \frac{\partial^j z_n(\boldsymbol{x}, t)}{\partial x_i^j}\right]',$$

$$z_{i\boldsymbol{x}}(\boldsymbol{x}, t) := \frac{\partial z_i(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} := \left[\frac{\partial z_i(\boldsymbol{x}, t)}{\partial x_1}, \cdots, \frac{\partial z_i(\boldsymbol{x}, t)}{\partial x_n}\right],$$

$$\nabla := \left[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right]',$$

$$\nabla^2 := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

$$\boldsymbol{z} \cdot \nabla := z_1 \frac{\partial}{\partial x_1} + \cdots + z_n \frac{\partial}{\partial x_n},$$

$$s(x_i) = \begin{cases} -1 & (x_i = 0) \\ 1 & (x_i \neq 0) \end{cases}$$

$$\int_{\Omega} (\cdot) d\boldsymbol{x} := \int_0^h \cdots \int_0^h \int_0^h (\cdot) dx_1 dx_2 \cdots dx_n,$$

$$\int_{\partial\Omega_i} (\cdot) d\boldsymbol{x} := \int_0^h \cdots \int_0^h \int_0^h \cdots \int_0^h (\cdot) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n.$$

First, we consider the explicit form of Burgers' equation described by

$$\frac{\partial \boldsymbol{z}}{\partial t} = \nu \nabla^2 \boldsymbol{z} - (\boldsymbol{z} \cdot \nabla)\boldsymbol{z}.$$

For the above system, a framework for designing model predictive controller subject to constraints has been proposed in [15]. Note that the following system can be considered as a generalized class of systems described by parabolic partial differential equations.

$$\frac{\partial \boldsymbol{z}}{\partial t} = \boldsymbol{F}\left(\boldsymbol{z}, \cdots, \boldsymbol{z}_{x_i^j}, \cdots\right)$$

For this system, a generalized framework for designing model predictive controller subject to constraints has been proposed in [14]. Consequently, model predictive control method for the explicit form of parabolic partial differential equations has been well established.

In recent decades, the control problems for implicit systems form of systems have drawn the considerable attention of many researchers due to extensive applications of differential algebraic equations in physical systems, singular systems, electrical networks, economic systems, and other areas. However, a framework for designing model predictive controller for implicit form of Burgers' equation has not yet been proposed. Thus, this paper examines the control problem of implicit form of Burgers' equation.

Next, we consider the implicit form of Burgers' equation, which can be represented by

$$\boldsymbol{E}\frac{\partial \boldsymbol{z}}{\partial t} = \nu \nabla^2 \boldsymbol{z} - (\boldsymbol{z} \cdot \nabla)\boldsymbol{z} \tag{1}$$

with the boundary condition

$$\frac{\partial \boldsymbol{z}(\boldsymbol{x}, t)}{\partial x_i} = \boldsymbol{G}_i(\boldsymbol{x})\boldsymbol{u}_i(\boldsymbol{x}, t) \text{ for } \boldsymbol{x} \in \partial\Omega_i \tag{2}$$

and the initial condition $\boldsymbol{z}(\boldsymbol{x}, 0) = \boldsymbol{z}_0(\boldsymbol{x})$, where $\boldsymbol{u}_i := [u_{i_1}, \cdots, u_{i_n}]' \in \mathbb{R}^n$ is the control input and $\boldsymbol{G}_i := \mathrm{diag}\{g_{i_1}, \cdots, g_{i_n}\} \in \mathbb{R}^{n \times n}$ is a space-dependent coefficient that is introduced to account for restrictions on the allocation of control actuators. For notational simplicity, we rewrite (1) as

$$\boldsymbol{E}\frac{\partial \boldsymbol{z}}{\partial t} = \boldsymbol{A}\left(\boldsymbol{z}, \boldsymbol{z}_{x_i^j}\right) := \nu \nabla^2 \boldsymbol{z} - (\boldsymbol{z} \cdot \nabla)\boldsymbol{z} \tag{3}$$

where $i = 1, \cdots, n$ and $j = 1,2$. Furthermore, we impose the following equality constraint on the control problem.

$$\boldsymbol{C}\left(\boldsymbol{u}_i, \boldsymbol{z}, \boldsymbol{z}_{x_i^j}\right) = 0 \tag{4}$$

where $\boldsymbol{C}$ is an $m-$dimensional vector-valued function. It is known that inequality constraints can be converted into equality constraints by introducing a slack variable.

Here, we impose an important assumption on systems (3).

[**Assumption 1**]
It is assumed that $\boldsymbol{E} \in \mathbb{R}^{n \times n}$ is not necessarily regular.

It is worth noting that if $\boldsymbol{E}$ is nonsingular that means invertible, then system (3) can be reduced to the explicit form of equation as follows:

$$\frac{\partial \boldsymbol{z}}{\partial t} = \boldsymbol{F}\left(\boldsymbol{z}, \boldsymbol{z}_{x_i^j}\right) = \boldsymbol{E}^{-1}\boldsymbol{A}\left(\boldsymbol{z}, \boldsymbol{z}_{x_i^j}\right)$$

Due to Assumption 1, implicit system (3) can be considered as a more generalized class of systems that include explicit systems.

III. FORMULATION OF OPTIMAL CONTROL PROBLEM

In this section, we consider the optimal control problem for implicit form of Burgers' equation described by (3). At each time t, the control input is determined to minimize the following evaluation function:

$$J = \int_{\Omega} \phi[\boldsymbol{z}(\boldsymbol{x}, t+T)]d\boldsymbol{x} + \int_{t}^{t+T}\int_{\Omega} L[\boldsymbol{z}(\boldsymbol{x}, \tau), \boldsymbol{u}(\boldsymbol{x}, \tau)]d\boldsymbol{x}d\tau \tag{5}$$

where $T \in \mathbb{R}_+$ is the evaluation interval of the performance index, $\phi \in \mathbb{R}_+$ is the terminal cost function, and $L \in \mathbb{R}_+$ is the stage cost function over the evaluation interval. In general, horizon T may vary with time; i.e., $T = T(t)$.

In this study, the evaluation function J is assumed to be a downward convex function. Thereby, the minimization of J can be reduced to finding the stationary point on the basis of the variational principle. Under this assumption, the stationary conditions that must be satisfied for the evaluation function J to be minimized are derived using the variational principle in the next section.

The optimization problem of (5) subject to (3) and (4) can be reduced to the minimization of the following performance index introduced using the costate $\boldsymbol{\lambda}(\boldsymbol{x}, t): \Omega \times \mathbb{R}_+ \to \mathbb{R}^n$ and the Lagrange multiplier $\boldsymbol{\mu}(\boldsymbol{x}, t)$: $\Omega \times \mathbb{R}_+ \to \mathbb{R}^n$ associated with the Burgers' equation (3), and the equality constraint (4), respectively.

$$\bar{J} = \int_{\Omega} \phi[\boldsymbol{z}(\boldsymbol{x}, t+T)]d\boldsymbol{x} + \int_{t}^{t+T}\int_{\Omega}\left(H - \lambda'(\boldsymbol{x}, \tau)\boldsymbol{E}\frac{\partial \boldsymbol{z}(\boldsymbol{x}, \tau)}{\partial \tau}\right)d\boldsymbol{x}d\tau \tag{6}$$

Therein, $H \in \mathbb{R}$ denotes the Hamiltonian defined by

$$H\left(\boldsymbol{z}, \boldsymbol{z}_{x_i^j}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}\right) := L + \boldsymbol{\lambda}'\boldsymbol{A} + \boldsymbol{\mu}'\boldsymbol{C}.$$

It is worth noting that the optimal control problem of implicit form of system (3) can be reduced to finding the control input that minimizes the performance index $\bar{J}$ in (6). In the case of optimal control problem for explicit form of Burgers' equation, the corresponding $\bar{J}$ is differently given as follows:

$$\bar{J} = \int_{\Omega} \phi[\boldsymbol{z}(\boldsymbol{x}, t+T)]d\boldsymbol{x} + \int_{t}^{t+T}\int_{\Omega}\left(H - \lambda'(\boldsymbol{x}, \tau)\frac{\partial \boldsymbol{z}(\boldsymbol{x}, \tau)}{\partial \tau}\right)d\boldsymbol{x}d\tau$$

Hereafter, we consider the minimization problem of the performance index $\bar{J}$ in (6) different from the above $\bar{J}$. It is well known that if $\bar{J}$ is minimized, then $\bar{J}$ takes a stationary value. In the next section, we consider the stationary conditions that must be satisfied for the performance index $\bar{J}$ to be minimized.

IV. NECESSARY CONDITIONS FOR OPTIMIZATION

In this section, we derive necessary conditions for the performance index $\bar{J}$ in (6) to be minimized. $\bar{J}$ takes a stationary value when $\bar{J}$ is minimized. Let the variation, also considered as infinitesimal change, in $\bar{J}$ be denoted by $\delta\bar{J}$. It is known that $\delta\bar{J} = 0$ must be satisfied for the performance index $\bar{J}$ in (6) to be minimized.

Here, let $\delta\boldsymbol{z}, \delta\boldsymbol{z}_\tau, \delta\boldsymbol{z}_{x_i^j}, \delta\boldsymbol{u}, \delta\boldsymbol{\lambda}$ and $\delta\boldsymbol{\mu}$ denote the variations in $\boldsymbol{z}, \boldsymbol{z}_\tau, \boldsymbol{z}_{x_i^j}, \boldsymbol{u}, \boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, respectively. Note that $\delta\bar{J}$ is depending on $\delta\boldsymbol{z}, \delta\boldsymbol{z}_\tau, \delta\boldsymbol{z}_{x_i^j}, \delta\boldsymbol{u}, \delta\boldsymbol{\lambda}$ and $\delta\boldsymbol{\mu}$. If $\delta\bar{J} = 0$ is satisfied, then each coefficient of $\delta\boldsymbol{z}, \delta\boldsymbol{z}_\tau, \delta\boldsymbol{z}_{x_i^j}, \delta\boldsymbol{u}, \delta\boldsymbol{\lambda}$ and $\delta\boldsymbol{\mu}$ must be zero. Those conditions are called the stationary conditions. Therefore, the stationary conditions are necessary conditions that must be satisfied for the performance index $\bar{J}$ in (6) to be minimized. In the following, we derive the stationary condition using the variational principle. For this purpose, we compute the variation $\delta\bar{J}$ caused by $\delta\boldsymbol{z}, \delta\boldsymbol{z}_\tau, \delta\boldsymbol{z}_{x_i^j}, \delta\boldsymbol{u}, \delta\boldsymbol{\lambda}$ and $\delta\boldsymbol{\mu}$.

When we focus on the only the relation between $\delta\bar{J}$ and $\delta\boldsymbol{z}$, we have the following equation.

$$\delta\bar{J} = \bar{J}(\boldsymbol{z} + \delta\boldsymbol{z}) - \bar{J}(\boldsymbol{z})$$

Applying the Taylor expansion to the first term on the right-hand side of the equation and neglecting the second-order and higher terms of $\delta\boldsymbol{z}$, we obtain the following equation.

$$\delta\bar{J} = \frac{\partial\bar{J}}{\partial\boldsymbol{z}}\delta\boldsymbol{z}$$

Similar to the above calculation, we can obtain the relation between $\delta\bar{J}$ and all variations $\delta\boldsymbol{z}, \delta\boldsymbol{z}_\tau, \delta\boldsymbol{z}_{x_i^j}, \delta\boldsymbol{u}, \delta\boldsymbol{\lambda}$ and $\delta\boldsymbol{\mu}$.

$$\begin{aligned}\delta\bar{J} = {} & \frac{\partial\bar{J}}{\partial\boldsymbol{z}(\boldsymbol{x}, t+T)}\delta\boldsymbol{z}(\boldsymbol{x}, t+T) + \frac{\partial\bar{J}}{\partial\boldsymbol{z}}\delta\boldsymbol{z} + \frac{\partial\bar{J}}{\partial\boldsymbol{z}_\tau}\delta\boldsymbol{z}_\tau \\ & + \frac{\partial\bar{J}}{\partial\boldsymbol{z}_{x_i^j}}\delta\boldsymbol{z}_{x_i^j} + \frac{\partial\bar{J}}{\partial\boldsymbol{z}_{\boldsymbol{u}}}\delta\boldsymbol{z}_{\boldsymbol{u}} + \frac{\partial\bar{J}}{\partial\boldsymbol{\lambda}}\delta\boldsymbol{\lambda} \\ & + \frac{\partial\bar{J}}{\partial\boldsymbol{\mu}}\delta\boldsymbol{\mu}\end{aligned} \tag{7}$$

Here, we focus on the third term on the right-hand side of (7). Then, applying the following integration by parts into that term, we have the following equation:

$$\frac{\partial\bar{J}(\boldsymbol{z})}{\partial\boldsymbol{z}_\tau}\delta\boldsymbol{z}_\tau = \int_{t}^{t+T}\int_{\Omega} -\boldsymbol{\lambda}'(\boldsymbol{x}, \tau)\boldsymbol{E}\frac{\partial\delta\boldsymbol{z}(\boldsymbol{x}, \tau)}{\partial\tau}\,d\boldsymbol{x}d\tau$$

$$
\begin{aligned}
&= \left[\int_{\Omega} -\boldsymbol{\lambda}'(\boldsymbol{x},\tau)\boldsymbol{E}\delta\boldsymbol{z}(\boldsymbol{x},\tau)\, d\boldsymbol{x}\right]_t^{t+T} \\
&\quad + \int_t^{t+T}\int_{\Omega}\left(\frac{\partial\boldsymbol{\lambda}(\boldsymbol{x},\tau)}{\partial\tau}\right)' \boldsymbol{E}\delta\boldsymbol{z}(\boldsymbol{x},\tau)\, d\boldsymbol{x}d\tau \\
&= \int_{\Omega} -\boldsymbol{\lambda}'(\boldsymbol{x},t+T)\boldsymbol{E}\delta\boldsymbol{z}(\boldsymbol{x},t+T)\, d\boldsymbol{x} \\
&\quad + \int_t^{t+T}\int_{\Omega}\left(\frac{\partial\boldsymbol{\lambda}(\boldsymbol{x},\tau)}{\partial\tau}\right)' \boldsymbol{E}\delta\boldsymbol{z}(\boldsymbol{x},\tau)\, d\boldsymbol{x}d\tau
\end{aligned} \tag{8}
$$

In (8), we set $\delta\boldsymbol{z}(\boldsymbol{x},t) = 0$ because $\boldsymbol{z}(\boldsymbol{x},\tau)$ is fixed at $\tau = t$ as the current state. Using (8), $\delta\boldsymbol{z}_\tau$ can be converted into $\delta\boldsymbol{z}$.

Next, we focus on the fourth term on the right-hand side of (7). Using the integration by parts into this term, the variation $\delta\boldsymbol{z}_{x_i^j}$ can be converted into the variation $\delta\boldsymbol{z}$.

$$
\begin{aligned}
\frac{\partial\bar{J}}{\partial\boldsymbol{z}_{x_i}}\delta\boldsymbol{z}_{x_i} &= \int_t^{t+T}\int_{\Omega}\frac{\partial H}{\partial\boldsymbol{z}_{x_i}}\delta\boldsymbol{z}_{x_i}d\boldsymbol{x}d\tau \\
&= \int_t^{t+T}\int_{\partial\Omega_i}\left[\frac{\partial H}{\partial\boldsymbol{z}_{x_i}}\delta\boldsymbol{z}\right]_0^h d\boldsymbol{x}d\tau \\
&\quad - \int_t^{t+T}\int_{\Omega}\frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial\boldsymbol{z}_{x_i}}\right)\delta\boldsymbol{z}d\boldsymbol{x}d\tau
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\frac{\partial\bar{J}}{\partial\boldsymbol{z}_{x_i^2}}\delta\boldsymbol{z}_{x_i^2} &= \int_t^{t+T}\int_{\Omega}\frac{\partial H}{\partial\boldsymbol{z}_{x_i^2}}\delta\boldsymbol{z}_{x_i^2}d\boldsymbol{x}d\tau \\
&= \int_t^{t+T}\int_{\partial\Omega_i}\left\{\left[\frac{\partial H}{\partial\boldsymbol{z}_{x_i^2}}\delta\boldsymbol{z}_{x_i}\right]_0^h\right. \\
&\quad \left. - \left[\frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial\boldsymbol{z}_{x_i^2}}\right)\delta\boldsymbol{z}\right]_0^h\right\} d\boldsymbol{x}d\tau \\
&\quad + \int_t^{t+T}\int_{\Omega}\frac{\partial^2}{\partial x_i^2}\left(\frac{\partial H}{\partial\boldsymbol{z}_{x_i^2}}\right)\delta\boldsymbol{z}d\boldsymbol{x}d\tau
\end{aligned} \tag{10}
$$

From the boundary condition in (2), it follows that

$$
\delta\boldsymbol{z}_{x_i}(\boldsymbol{x},t) = \boldsymbol{G}_i(\boldsymbol{x})\delta\boldsymbol{u}_i(\boldsymbol{x},t). \tag{11}
$$

By substituting (11) into (10) for $\boldsymbol{x} \in \partial\Omega_i$, we obtain

$$
\begin{aligned}
&\int_t^{t+T}\int_{\Omega}\frac{\partial H}{\partial\boldsymbol{z}_{x_i^2}}\delta\boldsymbol{z}_{x_i^2}d\boldsymbol{x}d\tau \\
&= \int_t^{t+T}\int_{\partial\Omega_i}\left\{\left[\frac{\partial H}{\partial\boldsymbol{z}_{x_i^2}}\boldsymbol{G}_i(x)\delta\boldsymbol{u}_i\right]_0^h\right. \\
&\quad \left. - \left[\frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial\boldsymbol{z}_{x_i^2}}\right)\delta\boldsymbol{z}\right]_0^h\right\} d\boldsymbol{x}d\tau \\
&\quad + \int_t^{t+T}\int_{\Omega}\frac{\partial^2}{\partial x_i^2}\left(\frac{\partial H}{\partial\boldsymbol{z}_{x_i^2}}\right)\delta\boldsymbol{z}d\boldsymbol{x}d\tau
\end{aligned} \tag{12}
$$

Using the integration by parts in (8), (9), (12), we can compute $\delta\bar{J}$ as follow:

$$
\begin{aligned}
\delta\bar{J} &= \int_{\Omega}\left\{\frac{\partial\phi[\mathbf{z}(\mathbf{x},t+T)]}{\partial\mathbf{z}(\mathbf{x},t+T)} - \boldsymbol{\lambda}'(\mathbf{x},t+T)\mathbf{E}\right\}\delta\mathbf{z}(\mathbf{x},t+T)d\mathbf{x} \\
&\quad + \int_t^{t+T}\left[\int_{\Omega}\left[\delta\boldsymbol{\lambda}'\left(\mathbf{A}\left(\mathbf{z},\mathbf{z}_{x_i^j}\right) - \mathbf{E}\frac{\partial\mathbf{z}}{\partial\tau}\right) + \delta\boldsymbol{\mu}'\mathbf{C}\right.\right. \\
&\quad + \left\{\left(\frac{\partial\boldsymbol{\lambda}}{\partial\tau}\right)'\mathbf{E} + \frac{\partial H}{\partial\mathbf{z}}\right. \\
&\quad \left.\left. + \sum_{i=1}^{n}\sum_{j=1}^{2}(-1)^j\frac{\partial^j}{\partial x_i^j}\left(\frac{\partial H}{\partial\mathbf{z}_{x_i^j}}\right)\right\}\delta\mathbf{z}\right]d\mathbf{x} \\
&\quad + \sum_{i=1}^{n}\int_{\partial\Omega_i}\sum_{x_i\in\{0,h\}}\left[s(x_i)\left\{\sum_{j=1}^{2}(-1)^{j-1}\frac{\partial^{j-1}}{\partial x_i^{j-1}}\left(\frac{\partial H}{\partial\mathbf{z}_{x_i^j}}\right)\right\}\delta\mathbf{z}\right. \\
&\quad \left.\left. + \left\{\left(\frac{\partial H}{\partial\mathbf{u}_i}\right) + s(x_i)\left(\frac{\partial H}{\partial\mathbf{z}_{x_i^2}}\right)\mathbf{G}_i\right\}\delta\mathbf{u}_i\right]d\mathbf{x}\right]d\tau
\end{aligned} \tag{13}
$$

It is worth noting that the variations $\delta\boldsymbol{z}_\tau, \delta\boldsymbol{z}_{x_i^j}$ are transformed to the variations $\delta\boldsymbol{z}, \delta\boldsymbol{u}$ using the integration by parts. Therefore, it can be seen from Eq. (13) that $\delta\bar{J}$ consists of the terms of variations $\delta\boldsymbol{z}, \delta\boldsymbol{u}, \delta\boldsymbol{\lambda}$and $\delta\boldsymbol{\mu}$. Consequently, we see that if $\delta\bar{J} = 0$is satisfied, then each coefficient of $\delta\boldsymbol{z}, \delta\boldsymbol{u}, \delta\boldsymbol{\lambda}$ and $\delta\boldsymbol{\mu}$. must be zero.

On the basis of the variational principle, we obtain the necessary conditions for a stationary value of $\bar{J}$ over the evaluation interval ($t \le \tau \le t + T$) as follows. For $\boldsymbol{x} \in \Omega$, we have

$$
\boldsymbol{E}\frac{\partial\boldsymbol{z}(\boldsymbol{x},\tau)}{\partial\tau} = \boldsymbol{A}\left(\boldsymbol{z},\boldsymbol{z}_{x_i^j}\right) \tag{14}
$$

$$
\boldsymbol{\lambda}'(\boldsymbol{x},t+T)\boldsymbol{E} = \left\{\frac{\partial\phi[\boldsymbol{z}(\boldsymbol{x},t+T)]}{\partial\boldsymbol{z}(\boldsymbol{x},t+T)}\right\} \tag{15}
$$

$$
\begin{aligned}
\boldsymbol{E}'\left(\frac{\partial\boldsymbol{\lambda}(\boldsymbol{x},\tau)}{\partial\tau}\right) &= \left\{-\frac{\partial H}{\partial\boldsymbol{z}}\right. \\
&\quad \left. - \sum_{i=1}^{n}\sum_{j=1}^{2}(-1)^j\frac{\partial^j}{\partial x_i^j}\left(\frac{\partial H}{\partial\boldsymbol{z}_{x_i^j}}\right)\right\}'
\end{aligned} \tag{16}
$$

$$
\boldsymbol{C}\left(\boldsymbol{u}_i,\boldsymbol{z},\boldsymbol{z}_{x_i^j}\right) = 0 \tag{17}
$$

and for $\boldsymbol{x} \in \partial\Omega_i$ and $i = 1, \cdots, n$, we have

$$
\sum_{j=1}^{2}(-1)^{j-1}\frac{\partial^{j-1}}{\partial x_i^{j-1}}\left(\frac{\partial H}{\partial\boldsymbol{z}_{x_i^j}}\right) = 0 \tag{18}
$$

$$
\left(\frac{\partial H}{\partial\boldsymbol{u}_i}\right) + s(x_i)\left(\frac{\partial H}{\partial\boldsymbol{z}_{x_i^2}}\right)\boldsymbol{G}_i = 0. \tag{19}
$$

Equations (14)–(19) are necessary conditions that must be satisfied for $\bar{J}$ to be minimized. Those conditions are called as the stationary conditions. In other words, stationary conditions in (14)–(19) must be satisfied to minimize the performance index $\bar{J}$.

Next, let the obtained stationary conditions be checked in detail. First, note that (14) is the time-evolutionary equation of $\boldsymbol{z}(\boldsymbol{x},\tau)$ and it is corresponding to the implicit form of Burgers' equation in (1). This equation can be solved using the boundary condition in (2) and the initial condition $\boldsymbol{z}(\boldsymbol{x},0)=\boldsymbol{z}_0(\boldsymbol{x})$. Then, note that (15) denotes the relation between $\boldsymbol{z}(\boldsymbol{x},t+T)$ and $\boldsymbol{\lambda}(\boldsymbol{x},t+T)$. Thus, (15) is called the terminal conditions because the state and the costate at the terminal time are determined by this equation. Furthermore, note that (16) is the time-evolutionary equation of $\boldsymbol{\lambda}(\boldsymbol{x},\tau)$ and (18) is considered as the boundary conditions of $\boldsymbol{\lambda}(\boldsymbol{x},\tau)$. Note that the time-evolutionary equation of $\boldsymbol{\lambda}(\boldsymbol{x},\tau)$ in (16) can be solved backward from the terminal time to the current time using the terminal condition $\boldsymbol{\lambda}(\boldsymbol{x},t+T)$ and the boundary condition in (18). Then, (17) is the equality constraint imposed in (4). For a given initial state and a given candidate of the optimal control input, $\boldsymbol{z}(\boldsymbol{x},\tau)$ and $\boldsymbol{\lambda}(\boldsymbol{x},\tau)$ over the evaluation interval $(t\le\tau\le t+T)$ can be determined using (14)–(18). However, such $\boldsymbol{z}(\boldsymbol{x},\tau)$ and $\boldsymbol{\lambda}(\boldsymbol{x},\tau)$ do not necessarily satisfy (19) for a given initial state and a given candidate of optimal control input. If condition in (19) is not satisfied, then the candidate of optimal control input need to be updated to reduce the norm of (19). Thus, (19) is called the optimality condition because it can be used the optimality performance index.

On the one hand, the stationary conditions for the optimal control problem of the explicit form of Burgers' equation have been shown in [15]. In fact, Eqs. (17)-(19) are same as the stationary conditions derived in [15], but (14) and (16) are different from the stationary conditions derived in [15]. The time-evolutionary equation of $\boldsymbol{z}(\boldsymbol{x},\tau)$ and $\boldsymbol{\lambda}(\boldsymbol{x},\tau)$ and the terminal condition in [15] are shown below.

$$\frac{\partial\boldsymbol{z}(\boldsymbol{x},\tau)}{\partial\tau}=\boldsymbol{A}\left(\boldsymbol{z},\boldsymbol{z}_{x_i^j}\right)\tag{20}$$

$$\boldsymbol{\lambda}'(\boldsymbol{x},t+T)=\left\{\frac{\partial\phi[\boldsymbol{z}(\boldsymbol{x},t+T)]}{\partial\boldsymbol{z}(\boldsymbol{x},t+T)}\right\}\tag{21}$$

$$\left(\frac{\partial\boldsymbol{\lambda}(\boldsymbol{x},\tau)}{\partial\tau}\right)=\left\{-\frac{\partial H}{\partial\boldsymbol{z}}-\sum_{i=1}^{n}\sum_{j=1}^{2}(-1)^j\frac{\partial^j}{\partial x_i^j}\left(\frac{\partial H}{\partial\boldsymbol{z}_{x_i^j}}\right)\right\}'\tag{22}$$

Note that if $\boldsymbol{E}(\boldsymbol{z})$ is the identity matrix, then (14)–(16) are identical to (20)–(22).

A well-known difficulty in solving non-linear optimal control problems is that the obtained stationary conditions cannot be solved analytically in general. Therefore, several algorithms have been developed for numerically solving stationary conditions. Although we have analytically derived the exact stationary conditions, we need a numerical algorithm for solving the stationary conditions. To solve the stationary conditions using a numerical algorithm, we must discretize them into finite difference equations. In the following, we provide the discretized stationary conditions based on (14)–(19).

Let $\boldsymbol{x}\in\Omega$ be divided into n_x grid points, and let $\hat{\boldsymbol{x}}:=[\hat{x}_1,\cdots,\hat{x}_{n_x}]'\in\mathbb{R}^{n_x}$ denote the discretized spatial vector. All discretized variables of each $x_1,\cdots,x_n$ are unified into $\hat{\boldsymbol{x}}$. Likewise, let time $\tau\in[t,t+T]$ over the evaluation interval be divided into n_t steps, and let $\hat{\boldsymbol{\tau}}:=[\hat{\tau}_1,\cdots,\hat{\tau}_{n_t}]'\in\mathbb{R}^{n_t}$ denote the discretized temporal vector. Note that $\hat{\tau}_1$ is equal to the current time t. Let the set $\{\partial\hat{x}_{i,1},\cdots,\partial\hat{x}_{i,n_x}\}$ be given by $\{\hat{x}_1,\cdots,\hat{x}_{n_x}\}\cap\partial\Omega_i$.

Let $\partial\hat{\boldsymbol{x}}_i\in\mathbb{R}^{n_u}$ be defined by $\partial\hat{\boldsymbol{x}}_i:=[\partial\hat{x}_{i,1},\cdots,\partial\hat{x}_{i,n_x}]'$. Let $\hat{\boldsymbol{u}}(\partial\hat{\boldsymbol{x}},\hat{\boldsymbol{\tau}}):=[\hat{\boldsymbol{u}}_1(\partial\hat{\boldsymbol{x}}_1,\hat{\boldsymbol{\tau}}),\ \cdots,\hat{\boldsymbol{u}}_n(\partial\hat{\boldsymbol{x}}_n,\hat{\boldsymbol{\tau}})]'$ denote the discretized control input. Let $\hat{\boldsymbol{z}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{\tau}})$, $\hat{\boldsymbol{\lambda}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{\tau}})$ and $\hat{\boldsymbol{\mu}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{\tau}})$ denote the discretized state, costate and Lagrange multiplier, respectively. Note that $\boldsymbol{z}(\boldsymbol{x},\tau)$ and $\hat{\boldsymbol{z}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{\tau}})$ are n and $(n\cdot n_x\cdot n_t)$ dimensional vector-valued functions, respectively. For notational simplicity, let $\hat{\boldsymbol{z}}(\hat{\boldsymbol{x}},\hat{\tau}_k)$ be denoted by $\hat{\boldsymbol{z}}_k$ for $k=1,\cdots,n_t$. Note that $\hat{\boldsymbol{z}}_1\in\mathbb{R}^{n\cdot n_x}$ is equal to the current known state $\hat{\boldsymbol{z}}(\hat{\boldsymbol{x}},t)$. For other variables, we adopt a similar notation.

The finite difference approximation to (14)–(19) results in the following discretized stationary conditions over the evaluation interval $k=1,\cdots,n_t$. In detail, the forward difference scheme is applied to (23), while the backward difference scheme is applied to (25).

$$\hat{\boldsymbol{E}}\hat{\boldsymbol{z}}_{k+1}=\hat{\boldsymbol{A}}(\hat{\boldsymbol{z}}_k,\hat{\boldsymbol{u}}_k)\tag{23}$$

$$\hat{\boldsymbol{\lambda}}_{n_t}{}'\hat{\boldsymbol{E}}=\hat{\boldsymbol{\Phi}}(\hat{\boldsymbol{z}}_{n_t})\tag{24}$$

$$\hat{\boldsymbol{E}}'\hat{\boldsymbol{\lambda}}_k=\hat{\boldsymbol{D}}(\hat{\boldsymbol{\lambda}}_{k+1},\hat{\boldsymbol{z}}_k,\hat{\boldsymbol{z}}_{k+1},\hat{\boldsymbol{u}}_{k+1},\hat{\boldsymbol{\mu}}_{k+1})\tag{25}$$

$$\hat{\boldsymbol{C}}_k(\hat{\boldsymbol{u}}_k,\hat{\boldsymbol{z}}_k)=0\tag{26}$$

$$\hat{\boldsymbol{H}}(\hat{\boldsymbol{u}}_k,\hat{\boldsymbol{\mu}}_k,\hat{\boldsymbol{z}}_k,\hat{\boldsymbol{\lambda}}_k)=0\tag{27}$$

Here, $\hat{\boldsymbol{A}},\hat{\boldsymbol{\Phi}},\hat{\boldsymbol{D}},\hat{\boldsymbol{E}}\in\mathbb{R}^{n\cdot n_x}$, $\hat{\boldsymbol{C}}\in\mathbb{R}^{n_c}$ and $\hat{\boldsymbol{H}}\in\mathbb{R}^{n_e}$ denote appropriately given vector-valued functions, where n_c and n_e denote proper integers. The time-evolutionary equations of $\boldsymbol{z}$ and $\boldsymbol{\lambda}$ are discretized into a forward difference equation, (23) and a backward difference equation, (25), respectively. Note that the boundary condition of (2) is also discretized and applied in (23). Moreover, the equations obtained by discretizing (16) and (18) are unified into (25). The remaining stationary conditions of (17) and (19) are also discretized and are represented in general forms in (26) and (27), respectively.

V. CONCLUSION

In this study, we consider the optimal control problem for implicit form of Burgers' equation. First, we introduced the system model and some notations. Moreover, we formulated a framework for optimal control problem subject to equality constraints. Using the variational principle, we derived the stationary conditions that must be satisfied to optimize the control performance over the evaluation interval. Although we have analytically derived the exact stationary conditions, we need a numerical algorithm for solving the stationary conditions. To solve the stationary conditions using a numerical algorithm, we must discretize them into finite difference equations. Consequently, we derived the discretized stationary condition for the optimal control problem of implicit form of Burgers' equation. For the explicit form of Burgers' equation, a fast numerical algorithm based on the contraction mapping method [12] has been established in [15] for solving the discretized stationary condition. This algorithm can be modified to solve the discretized stationary conditions obtained for the optimal control problem of the implicit form of Burgers' equation. To develop a fast numerical algorithm for solving the discretized stationary condition obtained in this study is a future work.

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